# SHARP DE RHAM REALIZATION

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Abstract. We introduce the *sharp* (universal) extension of a 1-motive (with additive factors and torsion) over a field of characteristic zero. We define the sharp de Rham realization $T_\sharp$ by passing to the Lie-algebra. Over the complex numbers we then show a (sharp de Rham) comparison theorem in the category of formal Hodge structures. For a free 1-motive along with its Cartier dual we get a canonical connection on their sharp extensions yielding a perfect pairing on sharp realizations. We thus provide one-dimensional sharp de Rham cohomology $\mathrm{H}^1_{\sharp-\mathrm{dR}}$ of algebraic varieties.

## Introduction

Grothendieck's idea of $\natural$-extensions has been largely employed and exploited to various extents. In Deligne's construction (see [8], 10.1.7, *cf.* [15]) for any Deligne 1-motive $M$ over a field $k$, one obtains a universal $\mathbb{G}_a$-extension $M^\natural$ of $M$ by the vector group $\mathrm{Ext}(M, \mathbb{G}_a)^\vee$. This $M^\natural$ is a complex of $k$-group schemes which is no more a Deligne 1-motive. Passing to Lie algebras Deligne defined in *loc. cit.* the de Rham realization $T_{\mathrm{dR}}$ according with Grothendieck's description of one-dimensional de Rham cohomology of abelian varieties in characteristic zero and crystalline cohomology in positive characteristics. Actually, pushing these techniques much further, we can describe $\mathrm{H}^1_{\mathrm{dR}}$ of any (arbitrarily singular) algebraic scheme, as well as the (de Jong) $\mathrm{H}^1_{\mathrm{crys}}$ in positive characteristics, by means of universal $\mathbb{G}_a$-extensions of certain Picard 1-motives (and more, *e.g.*, see [1] for the full picture).

0.1. **Results.** We here deal with a *sharp* $\mathbb{G}_a$-extension, based on the named universal $\mathbb{G}_a$-extension but including $\mathbb{G}_a$-factors, *i.e.*, for Laumon 1-motives, and we show a "*new* $\mathrm{H}^1$" out of it. We actually work in the abelian category $\mathcal{M}_1^{\mathrm{a}}$ of 1-motives with torsion and additive factors, containing the category of Laumon 1-motives $\mathcal{M}_1^{\mathrm{a,fr}}$ as the Quillen exact subcategory of free objects (see §1 below for the definition, *cf.* [13], §5 and [4], §1).

. Firstly, we get the sharp $\mathbb{G}_a$-extension $M^\sharp = [F \xrightarrow{u^\sharp} G^\sharp]$ of a (effective Laumon) $k$-1-motive $M = [F \xrightarrow{u} G]$, over a field $k$ of characteristic zero, roughly as follows. We show that (see 2.1.7 and 2.2.7 below) as soon as $\mathrm{Hom}(M, \mathbb{G}_a) = 0$ the universal $\mathbb{G}_a$-extension $M^\natural$ exists. Denote by $M_\times$ the quotient of $M$ by the (maximal) $\mathbb{G}_a$-factor $V(G)$, *cf.* (2.1.2). The universal $\mathbb{G}_a$-extension $M_\times^\natural = [F \to G_\times^\natural]$ of $M_\times$ exists, see 2.2.3, and we have that if $M$ admits a universal $\mathbb{G}_a$-extension then $M^\natural = M_\times^\natural$. Note that, if $M$ is free and $M^* = [F' \to G']$ is the Cartier

dual, then the Lie algebra $\mathrm{Lie}\,F'^0$ of the connected formal group is dual to the underlying $k$-vector space $V(G)$. Moreover, if $M$ admits a universal extension then $M^\natural = M^\natural_\times$ is the Cartier dual of $[F'_{\text{ét}} \times \widehat{G'} \to G']$ where $F'_{\text{ét}}$ denotes the étale quotient and $\widehat{G'}$ the formal completion at the origin, see 2.2.10. Thus set $M^\sharp$ to be the Cartier dual of $\vec{M}^* := [F' \times \widehat{G}' \to G']$ and obtain $M^\sharp$ as an extension of $M$ by $\mathrm{Ext}(M_\times, \mathbb{G}_a)^\vee$, see (3.1.3): this $M^\sharp$ is just the pull-back of $M^\natural_\times$ along $M \to M_\times$ (see §3 for details).

. Secondly *via* $M^\sharp$ we obtain $T_\sharp$ passing to the Lie algebra, *i.e.*, see §3.2, define the *sharp de Rham* realization $T_\sharp(M) := \mathrm{Lie}\,(G^\sharp)$ in such a way that $T_\sharp$ can be extended to an *exact* functor from the abelian category $\mathcal{M}_1^{\mathrm{a}}$ to filtered $k$-vector spaces. We also show that $T_\sharp$ is compatible with Cartier duality, see §5. In fact, for $M$ free with Cartier dual $M'$ and $\mathcal{P} \in \mathrm{Biext}(M, M^*; \mathbb{G}_m)$ the Poincaré biextension, we get a canonical connection $\nabla^\sharp$ on the pull-back $\mathcal{P}^\sharp \in \mathrm{Biext}(M^\sharp, M^{*\sharp}; \mathbb{G}_m)$ of $\mathcal{P}$ yielding a *perfect* pairing $T_\sharp(M) \otimes T_\sharp(M^*) \to k$, see 5.3.5. Over $k = \mathbb{C}$ we then show a (de Rham) *comparison theorem*, see 4.4.8, saying that $T_\sharp(M)$ is the underlying $\mathbb{C}$-vector space of a formal Hodge structure associated to the $\sharp$-extension, *i.e.*, we provide a general formula $T_\oint(M^\sharp) \cong T_\oint(M)^\sharp$, by making the sharp construction working in the category of formal Hodge structures $\mathrm{FHS}_1$ and applying a suitable Hodge realization $T_\oint$ (see §4 below and *cf.* [2]).

. Thirdly and finally, see §6, the resulting $\mathrm{H}^1$'s of an algebraic $k$-scheme $X$ can be visualized *via* a Laumon 1-motive $\mathrm{Pic}^+_a(X)$ and, dually, the $\mathrm{H}_1$'s by its Cartier dual $\mathrm{Alb}^-_a(X)$, see [14] for details. For $k = \mathbb{C}$ we get the $\mathrm{H}^1_{\sharp-\mathrm{sing}}(X) := T_\oint(\mathrm{Pic}^+_a(X))$. For $k$ of characteristic zero we now can set $\mathrm{H}^1_{\sharp-\mathrm{dR}}(X) := T_\sharp(\mathrm{Pic}^+_a(X))$. Note that here we have set $\mathrm{Pic}^+_a(X) := [0 \to \mathrm{Pic}^0(X)]$ for $X$ proper over $k$. If $X$ is not proper, say smooth for simplicity, $\mathrm{Pic}^+_a(X)$ is given by $[F \to \mathrm{Pic}^0(\overline{X})]$ where $\overline{X}$ is a suitable compactification, $X = \overline{X} - Y$, the étale part $F_{\text{ét}}$ of the formal group $F$ is given by algebraically equivalent to zero divisors on $\overline{X}$ supported on $Y$ and $F^0$ has Lie algebra $\mathrm{H}^1_Y(\overline{X}, \mathcal{O}_{\overline{X}})$ modulo the image of $\mathrm{H}^0(X, \mathcal{O}_X)$: note that the Cartier dual $\mathrm{Alb}^-_a(X)$ is the maximal Faltings-Wüstholz [9] additive extension of the Serre's Albanese semi-abelian variety of $X$. Over $k = \mathbb{C}$, in §6.2 we also link $\mathrm{H}^1_{\sharp-\mathrm{dR}}(X)$ to the enriched Hodge structures of Bloch and Srinivas [7].

0.2. **Perspectives.** First of all we expect that a similar construction holds in positive characteristics providing a $T_\dagger$ and $\mathrm{H}^1_{\sharp-\mathrm{crys}}(X) := T_\dagger(\mathrm{Pic}^+_a(X))$ will be the sharp crystalline cohomology. For $k = \mathbb{C}$, the general plan is to associate to an algebraic $\mathbb{C}$-scheme $X$ a formal Hodge structure called "sharp" singular cohomology $\mathrm{H}^*_{\sharp-\mathrm{sing}}(X)$ which contains, in the underlying algebraic structure, a formal group which is an extension of ordinary singular cohomology, *i.e.*, $\mathrm{H}^*_{\sharp-\mathrm{sing}}(X)_{\text{ét}} = \mathrm{H}^*(X_{\mathrm{an}}, \mathbb{Z})$. There will be "sharp" versions of de Rham and crystalline as well as comparison theorems between them. Moreover, the largest 1-motivic part of $\mathrm{H}^{1+i}_{\sharp-\mathrm{sing}}(X)$, $\mathrm{H}^{1+i}_{\sharp-\mathrm{dR}}(X)$, etc. should be exactly that obtained by applying $T_\oint$, $T_\sharp$, etc. to an algebraically defined (effective Laumon) 1-motive

$\mathrm{Pic}^+_a(X,i)$ for $i \geq 0$ (generalizing Deligne's conjectures to 1-motives with additive factors, *cf.* [1]). The main goal of this paper is the first step in drawing such a picture for $\mathrm{H}^1$'s of these forthcoming "sharp" cohomologies in zero characteristic.

0.3. **Notations.** $k$ is a field of characteristic 0 and $\overline{k}$ is an algebraic closure. $\mathbf{Aff}/k$ is the category of affine $k$-schemes endowed with the fppf topology. $\mathbf{Ab}/k$ is the category of sheaves of abelian groups on $\mathbf{Aff}/k$. Given a free $k$-module $\mathcal{E}$ we denote by $\mathcal{E}^\vee$ the dual $\mathrm{Hom}(\mathcal{E},k)$. For a $k$-vector group $V$ we sometimes denote by $V$ also its Lie algebra. Given an algebraic $k$-group $G$ we denote by $\omega_G$ the $k$-module of invariant differentials on $G$ and by $\boldsymbol{\omega}_G$ the associated sheaf as well as the associated $k$-vector group. $\widehat{G}$ will denote the formal completion at the origin of $G$ and $\iota\colon \widehat{G} \hookrightarrow G$ the inclusion. Given a formal $k$-group $F$ we denote by $F^*$ its Cartier dual.

## 1. Laumon 1-motives

1.1. **Free 1-motives.** Recall that a *Laumon* $k$-1-*motive* or a *free* $k$-1-*motive* $M = [u\colon F \to G]$ is a two terms complex (in degree -1, 0) where $F$ is a formal $k$-group without torsion, $G$ is a connected algebraic $k$-group and $u$ is a morphism in $\mathbf{Ab}/k$ (*cf.* [13], 5.1.1). It is known that any formal $k$-group $F$ splits canonically as product $F^0 \times F_{\text{ét}}$ where $F^0$ is the identity component of $F$ and is a connected formal $k$-group, and $F_{\text{ét}} = F/F^0$ is étale. Moreover, $F_{\text{ét}}$ admits a maximal subgroup scheme $F_{\text{tor}}$, étale and finite, such that the quotient $F_{\text{ét}}/F_{\text{tor}} = F_{\text{fr}}$ is constant of the type $\mathbb{Z}^r$ over $\overline{k}$. One says that $F$ is without torsion if $F_{\text{tor}} = 0$. The group $G$ is extension of an abelian variety $A$ by a linear $k$-group $L$ that is product of its maximal subtorus $T$ with a vector $k$-group $V(G)$.

Morphisms of Laumon $k$-1-motives are morphisms as complexes. We will denote by $\mathcal{M}_1^{\mathrm{a,fr}}$ the category of Laumon $k$-1-motives.

1.1.1. **Proposition.** *The canonical functor* $\mathcal{M}_1^{\mathrm{a,fr}} \to D^b(\mathbf{Ab}/k)$ *is a full embedding into the derived category of bounded complexes of sheaves for the fppf topology on* $\mathbf{Aff}/k$.

*Proof.* The proof of this fact for Deligne 1-motives in [16], 2.3.1, works also for Laumon 1-motives. Indeed the vanishings $\mathrm{Hom}(G,F) = 0 = \mathrm{Ext}_{\mathbf{Ab}/k}(G,F)$ still hold because of A.4.4 & A.4.5. $\square$

1.2. **Cartier duality.** We recall here the definition of the Cartier dual of a free 1-motive $M = [u\colon F \to G]$. See also [13], 5.2.2. Denote by $M_A := M/L$ the 1-motive $[\overline{u}\colon F \to A]$ induced by $M$ *via* the projection $G \twoheadrightarrow A$. The Cartier dual of $M$ is the 1-motive $M^* := [u'\colon F' \to G']$ where

- $F'$ is the formal $k$-group Cartier dual of the affine algebraic $k$-group $L$.
- $G'$ is the algebraic $k$-group that represents the sheaf on $\mathbf{Ab}/k$

$$\underline{\mathrm{Ext}}(M_A,\mathbb{G}_m)\colon \quad S \rightsquigarrow \mathrm{Ext}_{C^{[-1,0]}(\mathbf{Ab}/k)}(M_A,\mathbb{G}_m) = \mathrm{Hom}_{D^b(\mathbf{Ab}/k)}(M_A,\mathbb{G}_m[1]);$$

- $u'\colon \underline{\mathrm{Hom}}(L,\mathbb{G}_m) \to \underline{\mathrm{Ext}}(M_A,\mathbb{G}_m)$ is the push-out morphism for the sequence

$$0 \to L \to M \to M_A \to 0. \tag{1.2.1}$$

We spend some words on the representability of $\underline{\mathrm{Ext}}(M_A,\mathbb{G}_m)$. Consider the sequence of Ext sheaves associated to

$$0 \to A \to M_A \to F[1] \to 0. \tag{1.2.2}$$

It provides

$$0 \to \underline{\mathrm{Hom}}(F,\mathbb{G}_m) \to \underline{\mathrm{Ext}}(M_A,\mathbb{G}_m) \to A' \xrightarrow{\rho} \underline{\mathrm{Ext}}(F,\mathbb{G}_m) \tag{1.2.3}$$

where $A' = \underline{\mathrm{Ext}}(A,\mathbb{G}_m)$ is the dual abelian variety of $A$. From Lemma A.4.6 $\rho = 0$, the sheaf $\underline{\mathrm{Ext}}(M_A,\mathbb{G}_m)$ is extension of $A'$ by an affine algebraic $k$-group and hence representable by an algebraic $k$-group.

1.3. **Exact sequences.** We will say that a sequence of two terms complexes of fppf-sheaves, *e.g.*, of free $k$-1-motives

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0 \tag{1.3.1}$$

is *strongly exact* if it is exact as sequence of complexes, *i.e.*, for $k$-1-motives, if the sequence of algebraic $k$-groups $0 \to G_1 \xrightarrow{f_G} G_2 \xrightarrow{g_G} G_3 \to 0$ is exact as well as the sequence of formal $k$-groups $0 \to F_1 \to F_2 \to F_3 \to 0$.[1] One can check that Cartier duality does not preserve in general strongly exact sequences. This is pointed out in [3] for Deligne 1-motives. For example, consider a non-trivial $l$-torsion point $a$ of an abelian variety $A$ for $l$ a prime number. It corresponds to a $\mathbb{G}_m$-extension $G'$ of $A'$. Moreover $G'$ is also extension

$$0 \to B' \to G' \xrightarrow{g} \mathbb{G}_m \to 0 \tag{1.3.2}$$

where $B'$ is an abelian variety isogenous to $A'$ and the composition of $g$ with the inclusion $\mathbb{G}_m \to G'$ is the $l$-multiplication. It is immediate to see that the dual sequence of (1.3.2) is not strongly exact (*cf.* [3], §1.8).

1.3.3. **Proposition.** *Let* $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ *be a strongly exact sequence of free* 1*-motives. Are equivalent:*

*i) the dual sequence is strongly exact;*

*ii) the complex* $\eta_L\colon 0 \to L_1 \xrightarrow{f_L} L_2 \xrightarrow{g_L} L_3 \to 0$ *is exact;*

*iii) the complex* $\eta_A\colon 0 \to A_1 \xrightarrow{f_A} A_2 \xrightarrow{g_A} A_3 \to 0$ *is exact.*

*Proof.* Expand (1.3.1) writing a diagram having the $0 \to L_i \to G_i \to A_i \to 0$ as vertical sequences. The exactness of $\eta_L$ , (resp. $\eta_A$) may fail only at $L_2$ (resp. at $A_1$). Indeed the cokernel of $g_L$ is a linear $k$-group. It is trivial, because quotient of the kernel of $g_A$. This implies also the exactness of $\eta_A$ at $A_2$. Now, it is immediate to check that the exactness of $\eta_L$ at $L_2$ is equivalent to the exactness of $\eta_A$ at $A_1$. Hence *ii)* $\Leftrightarrow$ *iii)*. Furthermore *i)* $\Rightarrow$ *ii)* because the dual sequence

[1] This is the definition of *exact* sequences in [13].

of $\eta_L$ is the sequence of formal groups. Conversely, *iii*) implies that the induced complex $0 \to M_{A_1} \to M_{A_2} \to M_{A_3} \to 0$ is strongly exact and hence, passing to duals, we get an exact sequence of algebraic $k$-groups $0 \to G_1' \to G_2' \to G_3' \to 0$; the complex of formal $k$-groups $F_i'$ is exact because of *ii*). □

1.3.4. **Remark.** If $M_1$ is a linear $k$-group or if $G_3 = 0$, the dual sequence is always strongly exact.

1.4. **1-motives with torsion.** The previous section 1.3 motivates the introduction of 1-motives with torsion as done in [4] for Deligne 1-motives.

1.4.1. **Definition.** An *effective* (Laumon) $k$-1-motive $M$ is a two terms complex (in degree $-1, 0$) $[u\colon F \to G]$ where $F$ is a formal $k$-group, $G$ is a connected algebraic $k$-group and $u$ is a morphism in $\mathbf{Ab}/k$. An *effective* morphism

$$f = (f_F, f_G)\colon [F_1 \stackrel{u_1}{\to} G_1] \to [F_2 \stackrel{u_2}{\to} G_2]$$

is a morphism of complexes. The corresponding category is denoted by $\mathcal{M}_1^{\mathrm{a,eff}}$.

An effective 1-motive $M$ is said to be *étale* (resp. *connected*, resp. *special*) if $F^0 = 0 = V(G)$ (resp. $F = F^0$ and $A = 0 = T$, resp. $F^0$ maps to $V(G)$ *via* $u$). Étale (resp. connected, resp. special) 1-motives are regarded as a full subcategory of $\mathcal{M}_1^{\mathrm{a,eff}}$.

Note that since the base field $k$ is assumed of zero characteristic we always have that $F_{\mathrm{tor}} \times_G V(G) = 0$. In particular, if $u$ is an isomorphism then $M = 0$.

Further, an effective map $f$ is a *quasi-isomorphism*, *i.e.*, $\mathrm{Ker}(u_1) \cong \mathrm{Ker}(u_2)$ and $\mathrm{Coker}(u_1) \cong \mathrm{Coker}(u_2)$, if and only if $f_G$ is an isogeny and $F_1 \cong G_1 \times_{G_2} F_2$; see [4] for the more classical étale case. Note that, since $\mathrm{char}(k) = 0$, we have that the isogeny $f_G : G_1 \to G_2$ is given by pulling back isogenies between their semi-abelian quotients; in fact, it is an isomorphism when restricted to the vector groups $V(G_1) \stackrel{\simeq}{\to} V(G_2)$.

For an effective 1-motive $M$ let

$$F_{\mathrm{tor}} \cap \mathrm{Ker}(u) := F_{\mathrm{tor}} \times_F \mathrm{Ker}(u) \quad \text{and} \quad u(F_{\mathrm{tor}}) := F_{\mathrm{tor}}/F_{\mathrm{tor}} \cap \mathrm{Ker}(u) \subseteq G.$$

We denote

- $M_{\mathrm{tor}} := [F_{\mathrm{tor}} \cap \mathrm{Ker}(u) \to 0]$, the torsion part of $M$;
- $M_{\mathrm{fr}} := [F/F_{\mathrm{tor}} \to G/u(F_{\mathrm{tor}})]$, the free part of $M$;
- $M_{\mathrm{tf}} := [F/F_{\mathrm{tor}} \cap \mathrm{Ker}(u) \to G]$ the torsion free part.

Note that all these operations leave untouched $F^0$ and $V(G)$. One says that $M$ is *torsion free* if $M_{\mathrm{tor}} = 0$; this does not imply that $F$ is torsion free! Note that $M \to M_{\mathrm{fr}}$ factors as

$$M \stackrel{\psi}{\to} M_{\mathrm{tf}} \stackrel{\phi}{\to} M_{\mathrm{fr}}$$

where $\phi$ is a quasi-isomorphism and we always have a strongly exact sequence

$$0 \to M_{\mathrm{tor}} \to M \stackrel{\psi}{\to} M_{\mathrm{tf}} \to 0$$

where $\psi$ is a strict epi-morphism. Recall that here, similarly to [4], §1, we say that an effective morphism $f$ is *strict* if $f_G$ has connected kernel.

1.4.2. **Lemma** (*cf.* [4], Prop. 1.3). *Let $f = (f_F, f_G) : M_1 \to M_2$ be an effective morphism. Then $f$ can be factored as follows*

$$\begin{tikzcd} M_1 \arrow[rr, "f"] \arrow[dr, "\widetilde{f}"'] & & M_2 \\ & \widetilde{M}_2 \arrow[ur] & \end{tikzcd} \tag{1.4.3}$$

*where $\widetilde{f}$ is a strict morphism and $\widetilde{M}_2 \to M_2$ is a quasi-isomorphism.*

*Proof.* It follows from *loc. cit.* by pulling back the corresponding isogeny of the semi-abelian scheme quotients. □

Furthermore the class of quasi-isomorphisms admits calculus of right fractions (*cf.* [4], Prop. 1.2, and [3], Appendix C). Define *morphisms* of 1-motives by inverting quasi-isomorphisms from the right, *i.e.*, a morphism is represented by $fg^{-1}$ where $f$ is effective and $g$ is a quasi-isomorphism.

1.4.4. **Definition.** Denote by $\mathcal{M}_1^{\mathrm{a}}$ (resp. $\mathcal{M}_1^{s}$) the category of 1-motives *with torsion* obtained localizing the category of effective Laumon (resp. special) 1-motives at the multiplicative class of quasi-isomorphisms.

1.4.5. **Proposition.** *$\mathcal{M}_1^{\mathrm{a}}$ is an abelian category. $\mathcal{M}_1^{\mathrm{a,fr}} \subset \mathcal{M}_1^{\mathrm{a}}$ is a Quillen exact sub-category such that $M \rightsquigarrow M_{\mathrm{fr}}$ is s left-adjoint to the embedding.*

*Proof.* Since (1.4.3) is granted the proof is similar to [4], §1, and the more detailed Appendix C in [3]. □

Note that $\mathcal{M}_1^{\mathrm{a,fr}} \subset \mathcal{M}_1^{\mathrm{a}}$ is providing $\mathcal{M}_1^{\mathrm{a,fr}}$ of an exact structure in such a way that the dual of (1.3.2) is exact. More generally, Cartier duality is exact but we won't make use of this fact so that we omit the proof. (One reduces itself easily to the étale case and *cf.* [3], §1.8.)

In the following we denote by $\mathcal{M}_1$ the category of étale 1-motives with torsion regarded as a full abelian sub-category of $\mathcal{M}_1^{\mathrm{a}}$ (remark that being étale is preserved by quasi-isomorphims). Deligne 1-motives $\mathcal{M}_1^{\mathrm{fr}}$ are free étale 1-motives. Similarly $\mathcal{M}_1^{s} \subset \mathcal{M}_1^{\mathrm{a}}$ is the full abelian sub-category of 1-motives with torsion that are special.

1.5. **Linearized 1-motives.** For the sake of exposition we introduce a category $\mathcal{M}_1^{\ell}$ which is equivalent to $\mathcal{M}_1^{\mathrm{a}}$ but where connected formal groups are replaced by $k$-vector spaces, since $\mathrm{char}(k) = 0$.

1.5.1. **Definition.** Let $\mathcal{M}_1^{\ell,\mathrm{eff}}$ be the category whose objects are pairs $(u_{\acute{e}t} \colon F_{\acute{e}t} \to G, u_a \colon F_a \to \mathrm{Lie}\,(G))$ with $G$ a connected algebraic $k$-group, $F_{\acute{e}t}$ an étale formal $k$-group, $u_{\acute{e}t}$ a morphism in $\mathbf{Ab}/k$ and $u_a$ a homomorphism of finite dimensional $k$-vector spaces. Effective morphisms are triples

$$(f_{\acute{e}t} \colon F_{\acute{e}t} \to F'_{\acute{e}t}, f \colon G \to G', f_a \colon F_a \to F'_a)$$

with $f_{\text{ét}}, f$ morphisms in $\mathbf{Ab}/k$ and $f_a$ a homomorphism of vector spaces such that the obvious squares commute. Let $\mathcal{M}_1^{\ell}$ be the category obtained by localizing from the right $\mathcal{M}_1^{\ell,\text{eff}}$ at the multiplicative class of quasi-isomorphisms on the first component (and isomorphisms on the second).

1.5.2. **Proposition.** *Let* $[u\colon F \to G]$ *be a* 1*-motive. The functor*

$$\mathcal{M}_1^{\text{a}} \to \mathcal{M}_1^{\ell}, \quad [u\colon F \to G] \mapsto \big(u_{\text{ét}}\colon F_{\text{ét}} \to G, u_a\colon \operatorname{Lie}(F^0) \xrightarrow{\operatorname{Lie}(u^0)} \operatorname{Lie}(G)\big)$$

*is an equivalence of categories.*

*Proof.* Given a pair $(u_{\text{ét}}\colon F_{\text{ét}} \to G, u_a\colon F_a \to \operatorname{Lie}(G))$ as above we get a connected formal $k$-group $F^0$ as the formal completion at the origin of the vector group $\operatorname{Spec}(k[F_a^{\vee}])$. Moreover, as $\widehat{G}$ is isomorphic to the formal completion at the origin of $\operatorname{Spec}(k[\operatorname{Lie}(G)^{\vee}])$ (*cf.* A.3) the homomorphism $u_a$ provides a morphism of formal $k$-groups $F^0 \to \widehat{G}$ and hence a morphism $F^0 \to G$ in $\mathbf{Ab}/k$. $\square$

The category $\mathcal{M}_1^{\ell}$ is somewhat meaningful in order to construct objects in $\mathcal{M}_1^{\text{a}}$ from geometric invariants associated to algebraic schemes (see [9] and [14]).

## 2. Universal extensions of 1-motives

Let $M = [u\colon F \to G]$ be an effective $k$-1-motive.

2.1. **Some notations.** For $M = [F \xrightarrow{u} G]$ over $k$ let $V(G) \subseteq G$ be the maximal vector subgroup of $G$ so that $G$ can be represented as follows

$$0 \to V(G) \to G \to G_{\times} \to 0 \tag{2.1.1}$$

where $G_{\times}$ is the semi-abelian quotient and $V(G) \cong \mathbb{G}_a^n$ for some $n$. Denote $u_{\times}$ the composition of $u$ and the projection $G \twoheadrightarrow G_{\times}$. Set

- $M_{\times} := [u_{\times}\colon F \to G_{\times}]$

in such a way that we have a short exact sequence of complexes

$$0 \to V(G)[0] \to M \to M_{\times} \to 0. \tag{2.1.2}$$

Moreover, recalling that $F = F^0 \times_k F_{\text{ét}}$ canonically, we denote by $u_{\text{ét}}$ the composition of $F_{\text{ét}} \hookrightarrow F$ and $u_{\times}$. Set

- $M_{\text{ét}} := [u_{\text{ét}}\colon F_{\text{ét}} \to G_{\times}]$.

It is an étale 1-motive; if $F_{\text{ét}}$ is free, $M_{\text{ét}}$ is a Deligne 1-motive. We further get a functor $M \rightsquigarrow M_{\text{ét}} : \mathcal{M}_1^{\text{a}} \to \mathcal{M}_1$, left inverse to the inclusion of étale 1-motives. We always have a strongly exact sequence

$$0 \to M_{\text{ét}} \to M_{\times} \to F^0[1] \to 0. \tag{2.1.3}$$

Given a connected algebraic $k$-group $G$ denote $\vec{G} := [\widehat{G} \xrightarrow{\iota} G]$ the induced 1-motive. Set moreover

- $\vec{M} := [\vec{u}\colon F \times \widehat{G} \to G]$

obtained as push-out of $M$ with respect to $G \to \vec{G}$. We also have the restriction of $\vec{M}$ to $\widehat{V(G)}$, *i.e.*, $[F \times \widehat{V(G)} \to G]$ which is also an extension of $M_\times$ by $\overrightarrow{V(G)}$. If $M$ is special, we can further set $M^0 := [F^0 \to V(G)]$, and then get

$$0 \to M^0 \to M \to M_{\text{ét}} \to 0 \tag{2.1.4}$$

so that $M \rightsquigarrow M_{\text{ét}} : \mathcal{M}_1^s \to \mathcal{M}_1$ is left adjoint to the inclusion $\mathcal{M}_1 \hookrightarrow \mathcal{M}_1^s$ (*cf.* [2, §2]).

2.1.5**. Definition.** Let $M$ be an effective $k$-1-motive such that $\mathrm{Hom}(M, W) = 0$ for any $k$-vector group $W$. We say that $M$ admits a *universal* $\mathbb{G}_a$*-extension* if it exists a $k$-vector group $\mathbb{V}(M)$ and an extension $M^\natural$ of $M$ by $\mathbb{V}(M)$ such that the push-out homomorphism

$$\mathrm{Hom}(\mathbb{V}(M), W) \longrightarrow \mathrm{Ext}(M, W) \tag{2.1.6}$$

is an isomorphism for any $k$-vector group $W$. It is immediate to check that $\mathbb{V}(M)$ has to be then the vector group associated to $\mathrm{Ext}(M, \mathbb{G}_a)^\vee$.

Observe that thanks to A.4.1 & A.4.2 the notation $\mathrm{Ext}(M, W)$ is not ambiguous.

2.1.7**. Remark.** Following [15], I, 1.7, one can see that $M$ admits a universal extension $M^\natural$ if and only if

a) $\mathrm{Hom}(M, \mathbb{G}_a) = 0$,
b) $\mathrm{Ext}(M, \mathbb{G}_a)$ is a $k$-vector space of finite dimension.

If $M^\natural$ exists, then $\mathrm{Hom}(M^\natural, \mathbb{G}_a) = 0 = \mathrm{Ext}(M^\natural, \mathbb{G}_a)$ and $M^{\natural\natural} = M^\natural$.

2.1.8**. Examples.** We have the following paradigmatic cases:

- $\mathbb{G}_a$ does not admit universal extension.
- $T^\natural = T$ for any $k$-torus $T$.
- For any abelian variety $A$ the universal extension $A^\natural$ exists (*cf.* [15]). As observed in [13], 5.2.5, $A^\natural$ is the Cartier dual of the 1-motive $\vec{A'} = [\widehat{A}' \to A']$.
- Any Deligne 1-motive $M_{\text{ét}} = [u_{\text{ét}} \colon F_{\text{ét}} \to G_\times]$ ($F_{\text{ét}}$ free) admits a universal $\mathbb{G}_a$-extension $M_{\text{ét}}^\natural = [u_{\text{ét}}^\natural \colon F_{\text{ét}} \to G_{\text{ét}}^\natural]$ (see [8]).

2.2. **Existence results.** We start showing that we can reduce to work with free 1-motives.

2.2.1**. Proposition.** *An effective* 1*-motive $M$ admits universal extension if and only if $M_{\text{fr}}$ does.*

*Proof.* Set $K := F_{\text{tor}} \cap \mathrm{Ker}(u)$ and consider the sequence

$$0 \to K[1] = M_{\text{tor}} \to M \to M_{\text{tf}} \to 0. \tag{2.2.2}$$

As $\mathrm{Hom}(M_{\text{tor}}, \mathbb{G}_a) = 0 = \mathrm{Ext}(M_{\text{tor}}, \mathbb{G}_a)$, conditions in 2.1.7 holds for $M$ if and only if they hold for $M_{\text{tf}}$. Moreover, if $M_{\text{tf}} = [F/K \xrightarrow{v} G^\natural]$ is the universal extension of $M_{\text{tf}}$, the universal extension of $M$ is $[F \to G^\natural]$ obtained *via* composition of $v$ with the canonical $F \to F/K$. As $M_{\text{tf}}$ and $M_{\text{fr}}$ are quasi-isomorphic, conditions

in 2.1.7 hold for both or none of them. Moreover if the universal extensions $M_{\mathrm{tf}}^{\natural}$ and $M_{\mathrm{fr}}^{\natural}$ exist, they are quasi-isomorphic and $M_{\mathrm{tf}}^{\natural}$ is obtained *via* pull-back of $M_{\mathrm{fr}}^{\natural}$ along $M_{\mathrm{tf}} \to M_{\mathrm{fr}}$. $\square$

We will see that for effective 1-motives, condition *a*) in 2.1.7 implies condition *b*). We start with the case $M = M_{\times}$.

2.2.3. **Proposition.** *Let $M$ be an effective 1-motive. The universal $\mathbb{G}_a$-extension*

$$M_{\times}^{\natural} = [u_{\times}^{\natural} : F \to G_{\times}^{\natural}]$$

*of $M_{\times}$ exists.*

*Proof.* By 2.2.1 we may suppose $F$ torsion free. As $G_{\times}$ is semi-abelian, $M_{\times}$ satisfies condition *a*) in 2.1.7; moreover, from (2.1.3), A.4.1 & A.4.6 we get

$$0 \to \mathrm{Hom}(F^0, \mathbb{G}_a) \to \mathrm{Ext}(M_{\times}, \mathbb{G}_a) \to \mathrm{Ext}(M_{\text{ét}}, \mathbb{G}_a) \to \mathrm{Ext}(F^0, \mathbb{G}_a) = 0.$$

Now, $M_{\text{ét}}$ is a Deligne 1-motive and hence $M_{\text{ét}}$ satisfies condition *b*) (*cf.* 2.1.8). As $\mathrm{Hom}(F^0, \mathbb{G}_a)$ is a free $k$-module (*cf.* A.4.3) also $M_{\times}$ satisfies condition *b*) and we are done. $\square$

As for Deligne 1-motives, we have the following description of $\mathrm{Ext}(M_{\times}, \mathbb{G}_a)^{\vee}$ in terms of invariant differentials:

2.2.4. **Lemma.** *Let $M^* = [u' : F' \to G']$ be the Cartier dual of $M$ free. Then*

$$\mathrm{Ext}(M_{\times}, \mathbb{G}_a)^{\vee} = \mathrm{Lie}\,(G')^{\vee} = \omega_{G'}. \tag{2.2.5}$$

*Moreover, $G_{\times}^{\natural}$ is the push-out of $A^{\natural}$ with respect to the canonical homomorphism*

$$\mathrm{Ext}(A', \mathbb{G}_a)^{\vee} = \boldsymbol{\omega}_{A'} \longrightarrow \boldsymbol{\omega}_{G'} = \mathrm{Ext}(M_{\times}, \mathbb{G}_a)^{\vee} = \mathrm{Ext}(M_A, \mathbb{G}_a)^{\vee}.$$

*Proof.* The arguments in [6], 2.6, work also for the effective 1-motive $M_{\times}$. The second assertion can be checked as for Deligne 1-motives (*cf.* [5]). $\square$

As $M$ is extension of $M_{\times}$ by the vector group $V(G)$, we may view $M$ as the push-out of the universal extension $M_{\times}^{\natural}$ of $M_{\times}$ with respect to a unique $v_M$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(M_{\times}, \mathbb{G}_a)^{\vee} & \longrightarrow & M_{\times}^{\natural} & \longrightarrow & M_{\times} & \longrightarrow & 0 \\ & & {\scriptstyle v_M}\downarrow & & {\scriptstyle v_M^{\natural}}\downarrow & & \|\downarrow & & \\ 0 & \longrightarrow & V(G) & \longrightarrow & M & \longrightarrow & M_{\times} & \longrightarrow & 0 \end{array} \tag{2.2.6}$$

2.2.7. **Proposition.** *Let $M = [u\colon F \to G]$ be an effective 1-motive. Are equivalent:*

*i*) *$M$ admits a universal $\mathbb{G}_a$-extension,*
*ii*) $\mathrm{Hom}(M, \mathbb{G}_a) = 0$,
*iii*) *$v_M$ is surjective.*

*Moreover, if $M$ admits a universal extension then $M^{\natural} = M_{\times}^{\natural}$.*

*Proof.* We start showing that $\mathrm{Hom}(M,\mathbb{G}_a)=0$ if and only if $v_M$ is surjective. From (2.1.2) we get a sequence

$$0\to \mathrm{Hom}(M,\mathbb{G}_a)\to \mathrm{Hom}(V(G),\mathbb{G}_a)\xrightarrow{\partial}\mathrm{Ext}(M_\times,\mathbb{G}_a)\to \mathrm{Ext}(M,\mathbb{G}_a)\to 0.$$

Now, $\mathrm{Hom}(M,\mathbb{G}_a)=0$ if and only if $\partial$ is injective. However, $\partial$ is the push-out homomorphism along $v_M$ followed by the isomorphism $\mathrm{Hom}(\mathrm{Ext}(M_\times,\mathbb{G}_a)^\vee,\mathbb{G}_a)=\mathrm{Ext}(M_\times,\mathbb{G}_a)$ (*cf.* (2.1.6)). Hence $\partial$ is injective if and only if the push-out along $v_M$ is injective and this last is equivalent to the surjectivity of $v_M$. Furthermore, $\partial$ coincides with the $\mathbb{G}_a$-dual map $v_M^\vee$ and hence we get the exact sequence

$$0\to \mathrm{Ext}(M,\mathbb{G}_a)^\vee\to \mathrm{Ext}(M_\times,\mathbb{G}_a)^\vee\xrightarrow{v_M} V(G)\to 0. \tag{2.2.8}$$

Moreover $i)\Rightarrow ii)$ by definition.

Suppose now that $v_M$ is surjective (an hence $\mathrm{Hom}(M,\mathbb{G}_a)=0$). From (2.2.8) and 2.2.3 we get that $\mathrm{Ext}(M,\mathbb{G}_a)^\vee=\mathrm{Ker}(v_M)$ is a finite-dimensional $k$-vector space. Hence $M$ clearly satisfies condition $b)$ and it admits a universal extension.

For the last assertion, note that $\mathrm{Ker}(v_M)=\mathrm{Ker}(v_M^\natural)$ (*cf.* (2.2.6) and (2.2.8)); it is immediate to check that

$$0\to \mathrm{Ext}(M,\mathbb{G}_a)^\vee\to M_\times^\natural\xrightarrow{v_M^\natural} M\to 0$$

satisfies the universal property. □

2.2.9. **Examples.** a) $\vec{\mathbb{G}}_a:=[\iota\colon \widehat{\mathbb{G}}_a\to\mathbb{G}_a]$, with $\iota$ the inclusion, is the universal extension of $\widehat{\mathbb{G}}_a[1]$. Note that the Cartier dual $\widehat{\mathbb{G}}_a[1]^*=\mathbb{G}_a$ does not admit a universal extension. More generally, let $F$ be a connected formal $k$-group. The universal extension of $F[1]$ is the 1-motive $\overrightarrow{\mathrm{Lie}\,(F)}=[F\xrightarrow{\iota}\mathrm{Lie}\,(F)]$. To show this fact, one uses

$$\mathrm{Ext}(F[1],\mathbb{G}_a)^\vee=\mathrm{Hom}(F,\mathbb{G}_a)^\vee=\mathrm{Hom}(\mathrm{Lie}\,(F),\mathbb{G}_a)^\vee=\mathrm{Lie}\,(F).$$

b) Let $F$ be a torsion formal $k$-group. Then $F[1]=F[1]^\natural$.

By making use of Laumon 1-motives one can give an interpretation of universal extensions in terms of dual 1-motives.

2.2.10. **Proposition.** *Let $M=[u\colon F\to G]$ be a free 1-motive and $M^*=[u'\colon F'\to G']$ its Cartier dual. If $M$ admits a universal extension $M^\natural=[F\to G^\natural]$ then $M^\natural=M_\times^\natural$ is the Cartier dual of $[(u',\iota)\colon F'_{\text{ét}}\times\widehat{G}'\to G']$ where*

$$(u',\iota)(x,y):=u'(x)+\iota(y).$$

*Proof.* We have already seen that $M^\natural=M_\times^\natural$. Hence we may reduce to the case $M=M_\times$. We start with the case $M=M_A=[F\to A]$. Let

$$0\to \boldsymbol{\omega}_{G'}\to E\to M_A\to 0$$

be the Cartier dual of $\vec{G'}=[\widehat{G}'\xrightarrow{\iota}G']$ (*cf.* A.3.1). Suppose given an extension $N$ of $M_A$ by a $k$-vector group $W$. The Cartier dual of $N$ is an extension of $W^*=\underline{\mathrm{Hom}}(W,\mathbb{G}_m)[1]$ by $G'$ (*cf.* 1.3.4), hence it corresponds to a morphism

$h\colon \underline{\mathrm{Hom}}(W,\mathbb{G}_m)\to G'$. As $\underline{\mathrm{Hom}}(W,\mathbb{G}_m)$ is a connected formal $k$-group, $h$ factors through a unique morphism $\overline{h}\colon \underline{\mathrm{Hom}}(W,\mathbb{G}_m)\to \widehat{G}'$ and hence $N$ is the push-out of $E$ *via* the dual morphism $h^*\colon \boldsymbol{\omega}_{G'}\to W$. In particular, $E$ is the universal $\mathbb{G}_a$-extension of $M_A$.

In the case $T$ is not trivial, $M_\times$ is extension of $M_A$ by $T$. Its universal extension is the pull-back of $M_A^\natural$ along $M_\times\to M_A$. Hence the Cartier dual of $M_\times^\natural$ is the push-out of $\vec{G}'$ along $G'\to[u'\colon F'_{\text{ét}}\to G']=(M_\times)^*$ and this last is the 1-motive $[(u',\iota)\colon F'_{\text{ét}}\times\widehat{G}'\to G']$. □

2.3. **Exact sequences.** It follows from 2.2.7 that given a strongly exact sequence of 1-motives

$$0\to M_1\to M\to M_2\to 0 \tag{2.3.1}$$

if $M$ admits universal extension, then the same does $M_2$ while $M_1$ may not admit universal extension. For example, consider the sequence

$$0\to\mathbb{G}_a\to\vec{\mathbb{G}}_a\to\widehat{\mathbb{G}}_a[1]\to 0.$$

Moreover, we have:

2.3.2. **Lemma.** *If $M_1, M_2$ admit universal extensions also $M$ admits universal extension and*

$$0\to M_1^\natural\to M^\natural\to M_2^\natural\to 0$$

*is strongly exact.*

*Proof.* This fact follows immediately from 2.2.10 if the dual of (2.3.1) is still strongly exact. In the general case one has to check that given an isogeny of abelian varieties $\varphi\colon A\to B$, a free formal $k$-group $F$ and a morphism $u\colon F\to A$, the universal extension of $[u\colon F\to A]$ is the pull-back *via* $\varphi$ of the universal extension of $[\varphi\circ u\colon F\to B]$. □

2.3.3. **Remark.** In particular, the sequence (2.1.3) provides an exact sequence

$$0\to M_{\text{ét}}^\natural\to M_\times^\natural\to\overrightarrow{\mathrm{Lie}\,(F^0)}\to 0.$$

## 3. Sharp de Rham realization of 1-motives

3.1. **Sharp (universal) extension.** Proposition 2.2.10 shows that the universal extension, when it exists, forgets the contribution of the $k$-vector group $V(G)$, *i.e.*, of the connected formal group $F'^0$ of the dual. We introduce then a more general object.

3.1.1. **Definition.** Let $M=[F\xrightarrow{u}G]$ be a free 1-motive and $M^*=[F'\xrightarrow{u'}G']$ its Cartier dual. The *sharp* $\mathbb{G}_a$*-extension* $M^\sharp:=[u^\sharp\colon F\to G^\sharp]$ of $M$ is the Cartier dual of the 1-motive $\vec{M}^*=[(u',\iota)\colon F'\times\widehat{G'}\to G']$.

3.1.2**. Lemma.** *The free* 1*-motive* $M^\sharp$ *fits in the following pull-back diagram*

(3.1.3)
$$\begin{tikzcd} & & V(G) \arrow[r, equal] \arrow[d] & V(G) \arrow[d] & \\ 0 \arrow[r] & (\widehat{G}')^* = \mathrm{Ext}(M_\times, \mathbb{G}_a)^\vee \arrow[r] \arrow[d, equal] & M^\sharp \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Ext}(M_\times, \mathbb{G}_a)^\vee \arrow[r] & M_\times^\natural \arrow[r] \arrow[ur, dotted, "v_M^\sharp"] & M_\times \arrow[r] & 0 \end{tikzcd}$$

*Moreover, the homomorphism* $v_M^\natural\colon M_\times^\natural \to M$ *of (2.2.6) provides a splitting of the vertical sequence in the middle.*

*Proof.* As $\vec{M}^*$ fits in the following (strongly exact) sequence
$$0 \to M^* \to \vec{M}^* \to \widehat{G}'[1] \to 0,$$
passing to duals, one gets the horizontal sequence in the middle of (3.1.3). The map $M^\sharp \to M_\times^\natural$ is the dual of the canonical morphism $[F'_{\text{ét}} \times \widehat{G}' \to G'] \to \vec{M}^*$. The last assertion is immediate. □

3.1.4**. Lemma.** *The algebraic k-group* $G^\sharp$ *fits in the following diagram*
$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}([F^{\text{ét}} \to A], \mathbb{G}_a)^\vee = \boldsymbol{\omega}_{A'} \arrow[r] \arrow[d, hook, "i"] & A^\natural \times_A G \arrow[r] \arrow[d, hook] & G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Ext}(M_\times, \mathbb{G}_a)^\vee = \boldsymbol{\omega}_{G'} \arrow[r] \arrow[d, "\overline{\tau}"] & G^\sharp \arrow[r, "\rho"] \arrow[d, "\tau"] & G \arrow[r] & 0 \\ & \boldsymbol{\omega}_{L'} \arrow[r, equal] & \boldsymbol{\omega}_{L'} & & \end{tikzcd}$$

*that generalizes the one in* [5] *for Deligne* 1*-motives.*

*Proof.* By construction $G_\times^\natural$ is the push-out of $A^\natural$ with respect to $\boldsymbol{\omega}_{A'} \longrightarrow \boldsymbol{\omega}_{G'}$ (see (2.2.4)) and $G^\sharp$ is the pull-back *via* $G \to A$ of $G_\times^\natural$. The previous diagram says that we can take first the pull-back and then the push-out. □

Lemma 3.1.2 provides an alternative definition of $M^\sharp$ for free 1-motives that can be extended to effective 1-motives.

3.1.5**. Definition.** Let $M$ be an effective 1-motive. Denote by $M^\sharp := [F \to G^\sharp]$ the pull-back of $M_\times^\natural$ along $M \to M_\times$ and call it *sharp* $\mathbb{G}_a$*-extension* of $M$. In particular (3.1.3) holds.

By definition, $(M_\times)^\sharp = M_\times^\natural$. However this equality does not hold for a general $M$ that amdits universal extension. For example, $\vec{\mathbb{G}}_a^\natural = \vec{\mathbb{G}}_a$ while $\vec{\mathbb{G}}_a^\sharp = [\widehat{\mathbb{G}}_a \to \mathbb{G}_a^2]$ with the diagonal embedding as morphism.

3.1.6. **Lemma.** *The functor* $(\ )^\sharp\colon \mathcal{M}_1^{\mathrm{a}} \to \mathcal{M}_1^{\mathrm{a}}$ *is exact.*

*Proof.* Suppose given a quasi-isomorphism $f\colon M_1 \to M_2$ of effective 1-motives. It induces a quasi-isomorphism $f_\times\colon M_{1\times} \to M_{2\times}$ and then a quasi-isomorphism $M_{1\times}^\natural \to M_{2\times}^\natural$. In particular, $f^\sharp\colon M_1^\sharp \to M_2^\sharp$ is a quasi-isomorphism. Any short exact sequence in $\mathcal{M}_1^{\mathrm{a}}$ is isomorphic to a strongly exact sequence of effective 1-motives. Hence we may restrict to work with strongly exact sequences of effective 1-motives as in 2.3.1. Arguments used in the proof of 1.3.3 say that the complex of linear $k$-subgroups is exact (not necessarily strongly) and it is fixed by the $(\ )^\sharp$ functor. Moreover

$$0 \to M_{1,A_1} \to M_A \to M_{2,A_2} \to 0 \tag{3.1.7}$$

is exact. We are then reduced to see that $(\ )^\sharp$ applied to (3.1.7) preserves exactness. Thanks to the horizontal sequence in the middle of (3.1.3), it is sufficient to check that

$$0 \to \mathrm{Ext}(M_{1,A_1}, \mathbb{G}_a)^\vee \to \mathrm{Ext}(M_A, \mathbb{G}_a)^\vee \to \mathrm{Ext}(M_{2,A_2}, \mathbb{G}_a)^\vee \to 0$$

is exact. Let $B$ be the kernel of $A \to A_2$; it is an abelian variety isogenous to $A_1$. From A.4.2 and the isomorphism $\mathrm{Ext}([F_1 \to B], \mathbb{G}_a) = \mathrm{Ext}(M_{1,A_1}, \mathbb{G}_a)$, we get the result. $\square$

3.2. **Sharp de Rham.** We now can set the following:

3.2.1. **Definition.** Let $M$ be an effective 1-motive. Its *sharp de Rham realization* is

$$T_\sharp(M) := \mathrm{Lie}\,(G^\sharp).$$

Observe that $\mathrm{Lie}\,(G^\sharp)$ contains $V(G)$ and $\mathrm{Ext}(M_\times, \mathbb{G}_a)^\vee$ in such a way that $\mathrm{Ext}(M_{\text{ét}}, \mathbb{G}_a)^\vee \subseteq \mathrm{Ext}(M_\times, \mathbb{G}_a)^\vee$ with quotient $\mathrm{Hom}(F^0, \mathbb{G}_a)^\vee$ and we clearly have that $\mathrm{Ext}(M_\times, \mathbb{G}_a)^\vee \cap V(G) = 0$. The diagram of Lie algebras of (3.1.3) yields

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & V(G) \arrow[r, "="] \arrow[d] & V(G) \arrow[d] & \\
0 \arrow[r] & \mathrm{Ext}(M_\times, \mathbb{G}_a)^\vee \arrow[r] \arrow[d, "||"'] & \mathrm{Lie}\,(G^\sharp) \arrow[r] \arrow[d] & \mathrm{Lie}\,(G) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ext}(M_\times, \mathbb{G}_a)^\vee \arrow[r] & \mathrm{Lie}\,(G_\times^\natural) \arrow[r] \arrow[d] & \mathrm{Lie}\,(G_\times) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

and provides for a free 1-motive $M$

$$0 \to T_{\mathrm{dR}}(M_{\text{ét}}) \to T_\sharp(M)/V(G) \to \mathrm{Hom}(F^0, \mathbb{G}_a)^\vee \to 0$$

that is

$$0 \to T_\sharp(M_{\text{ét}}) \to T_\sharp(M_\times) \to T_\sharp(F^0[1]) \to 0.$$

Sharp de Rham realization is compatible with (2.1.2) and (2.1.3).

3.2.2. **Proposition.** *The functor $T_\sharp$ behaves well passing to localization on quasi-isomorphisms and it provides an exact functor from $\mathcal{M}_1^{\mathrm{a}}$ to the category of (filtered) k-vector spaces*

$$T_\sharp\colon \mathcal{M}_1^{\mathrm{a}} \longrightarrow \mathcal{V}_k.$$

*Proof.* We have already seen in 3.1.6 that $(\ )^\sharp$ is an exact functor. Moreover, any quasi-isomorphism $M_1 \to M_2$ induces an isomorphism $\mathrm{Lie}\,(G_1) \to \mathrm{Lie}\,(G_2)$. The conclusion follows recalling that any exact sequence is represented by an effective exact sequence. □

3.2.3. **Remark.** It follows from the proof of 2.2.1 that $M^\sharp$ is the pull-back of $M_{\mathrm{fr}}^\sharp$ along the canonical morphism $M \to M_{\mathrm{fr}}$ and $T_\sharp(M) \cong T_\sharp(M_{\mathrm{fr}})$.

## 4. Hodge theory

In this section $k = \mathbb{C}$. Also assume that the mixed Hodge structures are graded polarizable and denote by $\mathrm{MHS}_1$ the category of those structures with possibly non-zero Hodge numbers in the set $\{(0,0),(-1,0),(0,-1),(-1,-1)\}$, *i.e.*, of level $\leq 1$. The key point in what follows (*cf.* A.1.1 and 1.5.2) is that working with a connected formal $\mathbb{C}$-group $F^0$ we can think of $F^0$ also as $\mathrm{Lie}\,(F^0)$, the associated $\mathbb{C}$-vector group $\mathrm{Spec}(\mathbb{C}[\mathrm{Lie}\,(F^0)^\vee])$ or just the underlying $\mathbb{C}$-vector space.

4.1. **Formal Hodge structures.** This section is based on [2]. A formal Hodge structure (of level $\leq 1$) is: (*i*) a formal $\mathbb{C}$-group $H = H^0 \times H_\mathbb{Z}$ such that $H_\mathbb{Z}$ admits a mixed Hodge structure $H_{\text{ét}} = (H_\mathbb{Z}, W_*, F^*_{Hodge}) \in \mathrm{MHS}_1$, (*ii*) a $\mathbb{C}$-vector space $V$ with a sub-space $V^0 \subseteq V$, (*iii*) a "group homomorphism" $v\colon H \to V$, (*i.e.*, a homomorphism of $\mathbb{C}$-vector spaces $v^0\colon \mathrm{Lie}\,(H^0) \to V$ and a homomorphism of abelian groups $v_\mathbb{Z}\colon H_\mathbb{Z} \to V$) (*iv*) a $\mathbb{C}$-isomorphism $\sigma\colon H_\mathbb{C}/F^0_{Hodge} \cong V/V^0$. Moreover if $c\colon H_\mathbb{Z} \to H_\mathbb{C}/F^0_{Hodge}$ is the canonical map and $pr\colon V \to V/V^0$ is the projection, we assume that the following square

$$\begin{array}{ccc} H_\mathbb{Z} & \xrightarrow{v_\mathbb{Z}} & V \\ c\big\downarrow & & pr\big\downarrow \\ H_\mathbb{C}/F^0_{Hodge} & \xrightarrow{\sigma} & V/V^0 \end{array} \tag{4.1.1}$$

commutes.

We denote by $V^1$ the sub-space of $V$ that is the pull-back of $\sigma(W_{-2}H_\mathbb{C})$ *via* $V \to V/V^0$. It holds $V^0 \subseteq V^1 \subseteq V$. We denote by $v_\mathbb{C}$ the $\mathbb{C}$-linear map $H_\mathbb{C} \to V$ induced by $v_\mathbb{Z}$.

Denote by $(H,V)$ for short a formal Hodge structure. A morphism between $(H,V)$ and $(H',V')$ is a morphism of formal groups $h\colon H \to H'$ and a $\mathbb{C}$-homomorphism $g\colon V \to V'$ that respects the above structures and conditions. Denote by $\mathrm{FHS}_1$ the category of formal Hodge structures and by $\mathrm{FHS}_1^{\mathrm{fr}}$ the full subcategory given by $(H,V)$ with $H$ free. A formal Hodge structure $(H,V) \in \mathrm{FHS}_1$ is said to be *special* (resp. *connected*) if $v(H^0)$ lies in $V^0$ (resp. $H_{\mathbb{Z}} = 0$). Denote by $\mathrm{FHS}_1^s$ the full subcategory of special structures and by $\mathrm{FHS}_1^0$ the full subcategory of connected structures. Set

$$\overrightarrow{(H,V)} := (H \times \widehat{V}, V) \quad \text{and} \quad \overrightarrow{(H,V)}_0 := (H \times \widehat{V}_0, V) \tag{4.1.2}$$

where the filtration on $V$ remains the same and the morphism $\vec{v}\colon H \times \widehat{V} \to V$ is induced by $v\colon H \to V$ and $V \xrightarrow{id} V$ (or the inclusion $V_0 \subset V$).

4.2. **Enriched Hodge structures.** Recall that an enriched Hodge structure (of level $\leq 1$) is a pair $(H_{\text{ét}}, U \xrightarrow{u} V)$ where $H_{\text{ét}} = (H_{\mathbb{Z}}, W_*, F^*_{Hodge}) \in \mathrm{MHS}_1$, $u$ is a $\mathbb{C}$-linear map of $\mathbb{C}$-vector spaces and there exists a commutative diagram

$$\begin{tikzcd} H_{\mathbb{C}} \arrow[r, "\rho"] & U \arrow[r, "\pi"] \arrow[d, "u"] & H_{\mathbb{C}} \arrow[d, "c"] \\ & V \arrow[r, "\pi_0"] & H_{\mathbb{C}}/F^0_{Hodge} \end{tikzcd} \tag{4.2.1}$$

where the composition of the upper arrows is the identity (*cf.* [7]). The category of such enriched Hodge structures is denoted by $\mathrm{EHS}_1$. It is clear that we have a functor

$$\mathrm{EHS}_1 \longrightarrow \mathrm{FHS}_1, \quad (H_{\text{ét}}, U \xrightarrow{u} V) \mapsto (H_{\mathbb{Z}} \times \widehat{\mathrm{Ker}(\pi)}, V) \tag{4.2.2}$$

where $\iota\colon \widehat{\mathrm{Ker}(\pi)} \hookrightarrow \mathrm{Ker}(\pi)$ is the completion at the origin and $v\colon H_{\mathbb{Z}} \times \mathrm{Ker}(\pi) \to V$ is obtained *via* $u$ by composition with the maps $H_{\mathbb{Z}} \to H_{\mathbb{C}} \to U$ and $\mathrm{Ker}(\pi) \to U$. The sub-space $V^0$ of $V$ is defined as the kernel of $\pi_0$; hence we get an isomorphism $\sigma\colon H_{\mathbb{C}}/F^0_{Hodge} \to V/V^0$ and the diagram (4.1.1) commutes by construction. Furthermore, by construction $v(\mathrm{Ker}(\pi))$ lies in $V^0$; hence $(H_{\mathbb{Z}} \times \widehat{\mathrm{Ker}(\pi)}, V)$ is special. We actually obtain:

4.2.3. **Proposition.** $\mathrm{EHS}_1$ *is equivalent to the full subcategory* $\mathrm{FHS}_1^s$ *of* $\mathrm{FHS}_1$.

*Proof.* Let $(H,V)$ in $\mathrm{FHS}_1$. Note that giving the map $v\colon H \to V$ is equivalent to give a map $u\colon \mathrm{Lie}\,(H^0) \oplus H_{\mathbb{C}} \to V$ of $\mathbb{C}$-vector spaces. If $(H,V)$ is special then $\mathrm{Lie}\,(H^0)$ is mapped to $V^0$ and $(H_{\text{ét}}, H_{\mathbb{C}} \oplus \mathrm{Lie}\,(H^0) \xrightarrow{u} V) \in \mathrm{EHS}_1$ since *via* (4.1.1) the following

$$\begin{tikzcd} H_{\mathbb{C}} \arrow[r, "\rho"] & H_{\mathbb{C}} \oplus \mathrm{Lie}\,(H^0) \arrow[r, "\pi"] \arrow[d, "u"] & H_{\mathbb{C}} \arrow[d, "c"] \\ & V \arrow[r, "\pi_0"] & H_{\mathbb{C}}/F^0_{Hodge} \end{tikzcd} \tag{4.2.4}$$

commutes. The functor form $\mathrm{FHS}_1^s$ to $\mathrm{EHS}_1$ just defined and the one in 4.2.2 are clearly mutually quasi-inverse. □

4.3. **Hodge realization.** Deligne's Hodge realization in [8] provides an equivalence $T_{Hodge}\colon \mathcal{M}_1^{\mathrm{fr}} \xrightarrow{\sim} \mathrm{MHS}_1^{\mathrm{fr}}$ between the category of Deligne 1-motives and the category of torsion free objects in $\mathrm{MHS}_1$. This equivalence has been generalized in [4] to an equivalence $\mathcal{M}_1 \xrightarrow{\sim} \mathrm{MHS}_1$ including torsion and in [2] to an equivalence $\mathcal{M}_1^{\mathrm{a,fr}} \xrightarrow{\sim} \mathrm{FHS}_1^{\mathrm{fr}}$ including additive factors. We can now further extend both equivalences to our context (see 1.4.4 for notations).

4.3.1. **Proposition.** *There is an equivalence of categories*

$$T_{\oint}\colon \mathcal{M}_1^{\mathrm{a}} \xrightarrow{\sim} \mathrm{FHS}_1, \quad M := [u\colon F \to G] \quad \mapsto T_{\oint}(M) := (T_{\oint}(F), \mathrm{Lie}\,(G))$$

*where $T_{\oint}(M)_{\text{ét}} = T_{Hodge}(M_{\text{ét}})$ and such that it induces an equivalence of categories*

$$T_{\oint}^s\colon \mathcal{M}_1^s \xrightarrow{\sim} \mathrm{EHS}_1, \quad [u\colon F \to G] \quad \mapsto (T_{\oint}(M)_{\text{ét}}, T_{\mathbb{C}}(F_{\text{ét}}) \oplus \mathrm{Lie}\,(F^0) \to \mathrm{Lie}\,(G))$$

*and further restricts to the equivalence*

$$T_{Hodge}\colon \mathcal{M}_1 \xrightarrow{\sim} \mathrm{MHS}_1 \quad M \quad \mapsto T_{Hodge}(M).$$

*Proof.* The functor $T_{\oint}$ on $\mathcal{M}_1^{\mathrm{a,fr}}$ is constructed in [2]. Recall that for $F$ free, the formal $k$-group $T_{\oint}(F)$ is the product $F^0 \times T_{\oint}(F_{\text{ét}})$ where the étale quotient $T_{\oint}(F_{\text{ét}})$ is the pull-back of $F_{\text{ét}} \to G$ along $\exp : \mathrm{Lie}\,(G) \to G$. Hence $T_{\oint}(F_{\text{ét}})$ is a free abelian group extension of $F_{\text{ét}}$ by $H_1(G)$. The morphism $v\colon T_{\oint}(F) \to \mathrm{Lie}\,(G)$ is then taken as $\mathrm{Lie}\,(u^0)\colon \mathrm{Lie}\,(F^0) \to \mathrm{Lie}\,(G)$ on the identity component and the homomorphism obtained *via* the pull-back construction on $T_{\oint}(F_{\text{ét}})$. The above definition of $T_{\oint}(F)$ makes sense also when $F$ is not free. One proceed then as done in [4], Prop. 1.5; in fact, one can check that $T_{\oint}(F)$ and $\mathrm{Lie}\,(G)$ are independent of the representative of $M$, *i.e.*, that a quasi-isomorphism $M_1 \to M_2$ induces an isomorphism $T_{\oint}(M_1) \cong T_{\oint}(M_2)$. □

4.4. **Sharp envelope.** The sharp $\mathbb{G}_a$-extension of (effective) 1-motives, has its counterpart in the category of formal Hodge structures. Define the functor

$$(\ )^\sharp : \mathrm{FHS}_1 \to \mathrm{FHS}_1 \quad (H, V) \mapsto (H, V^\sharp) \tag{4.4.1}$$

as follows. Let $H = H_{\mathbb{Z}} \times H^0 \xrightarrow{v} V$. Denote $\overline{v}\colon H \to V/V^0$, $c\colon H_{\mathbb{Z}} \to H_{\mathbb{C}}$, $\iota : H^0 \to \mathrm{Lie}\,(H^0)$ the identity map on Lie algebras, $\overline{v}_{\mathbb{C}} : H_{\mathbb{C}} \to V/V^0$ and $\overline{v}^0\colon H^0 \to V/V^0$. Define first

- $(V/V^0)^\sharp := H_{\mathbb{C}} \oplus \mathrm{Lie}\,(H^0)$;
- $\overline{v}^\sharp := (c, \iota)\colon H_{\mathbb{Z}} \times H^0 \to (V/V^0)^\sharp$;
- $(V/V^0)^{\sharp 0} := \mathrm{Ker}(\overline{v}_{\mathbb{C}}, \overline{v}^0)$.

Hence we have a diagram:

(4.4.2)
$$\begin{tikzcd}
0 \arrow[r] & F^0_{Hodge} \arrow[r] \arrow[d] & H_{\mathbb{C}} \arrow[r, "\overline{v}_{\mathbb{C}}"] \arrow[d, "\overline{v}^{\sharp}_{\mathbb{C}}"] & V/V^0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (V/V^0)^{\sharp 0} \arrow[r] \arrow[d] & (V/V^0)^{\sharp} \arrow[r, "(\overline{v}_{\mathbb{C}}, \overline{v}^0)"] \arrow[d] & V/V^0 \arrow[r] & 0 \\
 & \mathrm{Lie}\,(H^0) \arrow[r, equal] & \mathrm{Lie}\,(H^0) \arrow[u, bend left, "\overline{v}^{\sharp 0}"] \arrow[ur, dotted, "\overline{v}^0"'] & &
\end{tikzcd}$$

where the vertical sequence in the middle is canonically split by $\overline{v}^{\sharp 0} := (\mathrm{id} \oplus 0)$. Define now $V^{\sharp}$ by pull-back as follows:

- $V^{\sharp} := V \times_{V/V^0} (V/V^0)^{\sharp}$;
- $v^{\sharp} := (v^{\sharp}_{\mathbb{Z}}, v^{\sharp 0})\colon H_{\mathbb{Z}} \times H^0 \to V^{\sharp}$ induced by $\overline{v}^{\sharp}$ and $v$
- $V^{\sharp 0} := \mathrm{Ker}(V^{\sharp} \to (V/V^0)^{\sharp} \to V/V^0)$.

Actually $V^{\sharp}$ fits in the following diagram of $\mathbb{C}$-vector spaces

$$\begin{tikzcd}
0 \arrow[r] & F^0_{Hodge} \arrow[r] \arrow[d] & H_{\mathbb{C}} \times_{V/V^0} V \arrow[r] \arrow[d, "\alpha"] & V \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (V/V^0)^{\sharp 0} \arrow[r] \arrow[d] & V^{\sharp} \arrow[r, "u"] \arrow[d] & V \arrow[r] & 0 \\
 & \mathrm{Lie}\,(H^0) \arrow[r, equal] & \mathrm{Lie}\,(H^0) \arrow[u, bend left, "v^{\sharp 0}"] \arrow[ur, dotted, "v^0"'] & &
\end{tikzcd}$$

with the canonical splitting $v^{\sharp 0}$ of the vertical sequence in the middle. Note that the morphism $v^{\sharp}_{\mathbb{Z}}\colon H_{\mathbb{Z}} \to V^{\sharp}$ is the composition of $(c, v_{\mathbb{Z}})\colon H_{\mathbb{Z}} \to H_{\mathbb{C}} \times_{V/V^0} V$ with $\alpha$ and the commutativity of 4.1.1 holds by construction.

4.4.3. **Remark.** Note that, for $(H,V) \in \mathrm{FHS}_1$, the sharp envelope $(H,V)^{\sharp} \in \mathrm{FHS}_1$ is such that the canonical map $v^{\sharp}_{\mathbb{C}}\colon H_{\mathbb{C}} \to V^{\sharp}$ induced by $v^{\sharp}_{\mathbb{Z}}$ has a splitting $\pi : V^{\sharp} \to H_{\mathbb{C}}$ induced by $v^{\sharp 0}$. Observe that if $(H,V)$ is étale, *i.e.*, $H^0 = V^0 = 0$, we then have $(H,V) \cong (H_{\mathbb{Z}}, H_{\mathbb{C}}/F^0_{Hodge})$ is determined by the mixed Hodge structure (*cf.* [2]). We moreover get $(H,V)^{\sharp} \cong (H_{\mathbb{Z}}, H_{\mathbb{C}}/F^0_{Hodge})^{\sharp} = (H_{\mathbb{Z}}, H_{\mathbb{C}})$.

4.4.4. **Lemma.** $(H,V) \in \mathrm{FHS}^s_1$ *if and only if* $(H, V/V^0)^{\sharp} \in \mathrm{FHS}^s_1$ *if and only if the splitting* $\overline{v}^{\sharp 0}$ *induces a splitting of the left most vertical sequence in* (4.4.2).

*Proof.* By diagram (4.4.2) chase, *i.e.*, $\overline{v}^0 : H^0 \xrightarrow{\text{zero}} V/V^0$ if and only if $\overline{v}^{\sharp 0}(H^0) \subseteq (V/V^0)^{\sharp 0}$. □

4.4.5. **Remark.** Since the sharp envelope of a special formal Hodge structure is still special, we can consider *via* 4.2.3 the sharp envelope on the category of enriched Hodge structrures (see §4.2): it is the functor

$$(\ )_e^\sharp \colon \mathrm{EHS}_1 \to \mathrm{EHS}_1, \quad (H_{\text{ét}}, U \to V) \mapsto (H_{\text{ét}}, U \to V^\sharp).$$

Here $H_{\text{ét}}$ and $U = H_{\mathbb{C}} \times \mathrm{Lie}\,(H^0)$ correspond to $(H, V) \in \mathrm{FHS}_1^s$.

We further obtain a more sophisticated functor

$$(\ )_s^\sharp : \mathrm{FHS}_1^s \to \mathrm{EHS}_1 \quad (H, V) \mapsto (H, V)_s^\sharp := (H_{\text{ét}}, V^\sharp \to V) \tag{4.4.6}$$

where $(H, V)_s^\sharp$ is the enriched Hodge structure associated to $\overrightarrow{(H, V)}_0$ in (4.1.2). In fact, given a special structure along with its sharp envelope we are just left to get the splitting $\pi$ fitting in a commutative diagram

$$\begin{tikzcd}
H_{\mathbb{C}} \arrow[r, "v_C^\sharp"] & V^\sharp \arrow[r, "\pi"] \arrow[d, "u"'] & H_{\mathbb{C}} \arrow[d, "c"'] \\
& V \arrow[r, "\pi_0"] & H_{\mathbb{C}}/F^0_{Hodge}
\end{tikzcd}$$

where $u : V^\sharp \twoheadrightarrow V$ is the canonical projection and $\pi_0$ is the projection induced by $\sigma^{-1} : V/V^0 \cong H_{\mathbb{C}}/F^0_{Hodge}$. Note that by 4.4.4 and the construction of $V^\sharp$ we then get a natural splitting of all extensions as follows

$$\begin{tikzcd}
0 \arrow[r] & H_{\mathbb{C}} \times_{V/V^0} V \arrow[r, "\alpha"'] \arrow[d] & V^\sharp \arrow[r] \arrow[l, bend left] \arrow[d] \arrow[dl, dotted, "\pi"] & H^0 \arrow[r] \arrow[l, bend left, "v^{\sharp 0}"] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_{\mathbb{C}} \arrow[r] & (V/V^0)^\sharp \arrow[r] \arrow[l, bend left] & H^0 \arrow[r] \arrow[l, bend left] & 0 \\
0 \arrow[r] & F^0_{Hodge} \arrow[r] \arrow[u] & (V/V^0)^{\sharp 0} \arrow[r] \arrow[l, bend left] \arrow[u] & H^0 \arrow[r] \arrow[l, bend left] \arrow[u, equal] & 0
\end{tikzcd}$$

We obtain the claimed commutativity by diagram chase just considering that all these splittings are compatible.

4.4.7. **Remark.** Fix $H_{\text{ét}} = (H_{\mathbb{Z}}, W_*, F^*_{Hodge}) \in \mathrm{MHS}_1$ and $\pi_0 : V \to H_{\mathbb{C}}/F^0_{Hodge}$. Then any $(H_{\text{ét}}, U \to V) \in \mathrm{EHS}_1$ is clearly mapped to $(H_{\text{ét}}, H_{\mathbb{C}}^\sharp \to V)$ where $H_{\mathbb{C}}^\sharp$ is just the pull-back of $H_{\mathbb{C}}$ along $\pi_0$. Actually, for any $(H_{\text{ét}} \times H^0, V) \in \mathrm{FHS}_1$ with $V^0 = \mathrm{Ker}(\pi_0)$, we have that $(H_{\text{ét}}, H_{\mathbb{C}}^\sharp)$ is the sharp envelope of $(H_{\text{ét}}, V)$ since $H_{\mathbb{C}}^\sharp \cong H_{\mathbb{C}} \times_{V/V^0} V$ and $H_{\mathbb{C}}^\sharp \hookrightarrow V^\sharp$ (in particular, any $U \to V$ as above lifts to $V^\sharp$).

Now, in 3.2 we associated to any effective 1-motive $M$ the sharp extension $M^\sharp$ and the sharp de Rham realization $T_\sharp(M) = \mathrm{Lie}\,(G^\sharp)$. Moreover, we can apply

$T_\oint$ to $M^\sharp$ so that we have a diagram

$$\begin{tikzcd}
\mathcal{M}_1^{\mathrm{a}} \arrow[r, "T_\oint"] \arrow[d, "(\ )^\sharp"'] & \mathrm{FHS}_1 \arrow[d, "(\ )^\sharp"] \\
\mathcal{M}_1^{\mathrm{a}} \arrow[r, "T_\oint"] & \mathrm{FHS}_1
\end{tikzcd}$$

4.4.8. **Theorem.** *Let $M = [u\colon F \to G]$ be a $\mathbb{C}$-1-motive. The diagram above commutes (up to isomorphisms) so that*

$$T_\oint(M)^\sharp \cong T_\oint(M^\sharp) = (T_\oint(F), T_\sharp(M)).$$

*Proof.* It is sufficient to prove the commutativity on the categories $\mathcal{M}_1^{\mathrm{a,fr}}$ and $\mathrm{FHS}_1^{\mathrm{fr}}$ (*cf.* 3.2.3). Let then $M = [F \to G]$ be a Laumon 1-motive. We may assume $V(G) = 0 = V^0$. Indeed $M^\sharp$ is obtained *via* pull-back from $(M_\times)^\sharp = M_\times^\natural$ (*cf.* 3.1.2) and $(H, V)^\sharp$ is defined as the pull-back of $(H, V/V^0)^\sharp$. Let then $M = M_\times = [F \to G_\times]$ and $M^\sharp = [F \to G_\times^\natural]$ its universal extension. We have

$$T_\oint(M) = (H, V) = (F^0 \times T_\oint(F_{\text{ét}}), \mathrm{Lie}\,(G_\times)) \text{ with } V^0 = 0,$$

$$T_\oint(M^\sharp) = (F^0 \times T_\oint(F_{\text{ét}}), \mathrm{Lie}\,(G_\times^\natural))$$

and $T_\oint(F_{\text{ét}}) \to \mathrm{Lie}\,(G_\times)$ that factors through $\mathrm{Lie}\,(G_\times^\natural)$ because $G_\times^\natural$ is extension of $G_\times$ by a vector group. Denote by $[F_{\text{ét}} \to G_{\text{ét}}^\natural]$ the universal extension of $M_{\text{ét}}$. From 2.3.3 we get a push-out diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(M_{\text{ét}}, \mathbb{G}_a)^\vee \arrow[r] \arrow[d] & G_{\text{ét}}^\natural \arrow[r] \arrow[d] & G_\times \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ext}(M_\times, \mathbb{G}_a)^\vee \arrow[r] \arrow[d] & G_\times^\natural \arrow[r] \arrow[d] & G_\times \arrow[r] & 0 \\
 & \mathrm{Lie}\,(F^0) \arrow[r, equal] & \mathrm{Lie}\,(F^0) & &
\end{tikzcd}$$

Consider the associated diagram of Lie algebras and compare it with (4.4.2). Recalling that $\mathrm{Lie}\,(G_{\text{ét}}^\natural) \cong H_\mathbb{C}$, $V = \mathrm{Lie}\,(G_\times)$, $V^0 = 0$, $\mathrm{Lie}\,(H^0) = \mathrm{Lie}\,(F^0)$ we deduce that $V^\sharp \cong \mathrm{Lie}\,(G_\times^\natural) = T_\sharp(M_\times)$. □

4.4.9. **Remark.** The previous theorem generalizes to (effective) Laumon $\mathbb{C}$-1-motives the fact that $T_{\mathrm{dR}}(M) \cong T_\mathbb{Z}(M) \otimes_\mathbb{Z} \mathbb{C}$ for Deligne 1-motives.

## 5. Duality on sharp de Rham realizations

Let $M$ be a Laumon 1-motive, $M^*$ its dual and $\mathcal{P} \in \mathrm{Biext}(M, M^*; \mathbb{G}_m)$ the Poincaré biextension. We look for a canonical duality between $T_\sharp(M)$ and $T_\sharp(M^*)$ that generalizes Deligne's construction in [8] 10.2.7.3. In order to do this we need

to introduce a "canonical" connection on the biextension $\mathcal{P}^\sharp$ of $M^\sharp$ and $M^{*\sharp}$ by $\mathbb{G}_m$ given by the pull-back of $\mathcal{P}$.

5.1. **$\natural$-structures.** Let $N = [F \to E]$ be an extension of $M$ by an algebraic $k$-group $H$. A *$\natural$-structure* on $N$ is a $\natural$-structure on $E$ as in [8], 10.2.7.1. A *strong $\natural$-structure* on $N$ is a $\natural$-structure on $E$ such that its pull-back to $F$ is trivial; if $F = F_{\text{ét}}$ any $\natural$-structure is strong. We can characterize $G^\sharp$ as follows:

5.1.1. **Proposition.** *Let $M$ be a Laumon 1-motive. Then $G^\sharp$ is the group scheme that represents the pre-sheaf for the flat site on $k$*

$$\mathcal{F}\colon S/k \rightsquigarrow \left\{ \begin{array}{ll} (g,\nabla), & g\in G(S), \nabla \text{ a } \natural\text{-structure on the extension } \mathcal{P}_g \text{ of} \\ & M' \text{ by } \mathbb{G}_{m,S} \text{ induced by } g \end{array} \right\}.$$

*Proof.* By (3.1.4) we know that $G^\sharp$ is extension of $\boldsymbol{\omega}_{L'}$ by $A^\natural \times_A G$ and one proceeds as in [6] 3.10. □

5.2. **The canonical connection.** The identity on $G^\sharp$ provides *via* the functor $\mathcal{F}$ a pair $(\rho, \nabla_2^\sharp)$ where $\rho\colon G^\sharp \to G$ is the usual projection and $\nabla_2^\sharp$ is a $\natural$-structure on $\mathcal{P}_\rho$ the pull-back of $\mathcal{P}$ to $G^\sharp \times G'$ viewed as $\mathbb{G}_m$-extension of $G^\sharp$ over $G'$. The same the identity on $G'^\sharp$ provides a pair $(\rho', \nabla_1^\sharp)$ where $\nabla_1^\sharp$ is a a $\natural$-structure on $\mathcal{P}_{\rho'}$. As $\mathcal{P}^\sharp$ is the pull-back of $\mathcal{P}_\rho$ *via* $\rho'$ as well as the pull-back of $\mathcal{P}_{\rho'}$ *via* $\rho$ we define the *canonical connection* $\nabla^\sharp$ on $\mathcal{P}^\sharp$ as the sum of the (pull-back) of the connections $\nabla_i^\sharp$. If $M$ is a Deligne 1-motive, $\nabla^\sharp$ is the unique $\natural$-structure on $\mathcal{P}^\sharp = P^\natural$ in [8] 10.2.7.4.

5.2.1. **Example.** Let $F^0 = \mathrm{Spf}(k[[x]])$, $M = F^0[1]$, $M^* = F^{0*} = \mathrm{Spec}(k[y])$ and $\mathcal{P}$ the Poincaré biextension. It is the trivial $\mathbb{G}_m$-torsor on $F^{0*}$ together with the trivialization $\sigma\colon F^0 \otimes F^{0*} \to \mathbb{G}_m$ induced by Cartier duality. The pull-back $\mathcal{P}^\sharp$ of $\mathcal{P}$ to $([F^0 \to \boldsymbol{\omega}_{F^{0*}}], F^{0*})$ is the trivial biextension on $(\boldsymbol{\omega}_{F^{0*}}, F^{0*})$ together with the trivialization $\sigma$. The connection $\nabla_2^\sharp$ of the trivial $\mathbb{G}_m$-extension of $F^{0*}$ over $\boldsymbol{\omega}_{F^{0*}}$ is given by the invariant differential of $F^{0*}$ over $\boldsymbol{\omega}_{F^{0*}}$ associated to the identity map on $\boldsymbol{\omega}_{F^{0*}}$; hence $xdy$. The connection $\nabla_1^\sharp$ is associated to an invariant differentials of the 0 group over $F^{0*}$ hence is trivial. In particular $\nabla^\sharp$ is associated to $xdy$ on $\boldsymbol{\omega}_{F^{0*}} \times_k F^{0*}$.

Observe that also $xdy + ydx$ provides a bi-invariant connection on $\mathcal{P}^\sharp$ different from the canonical one. Hence we can not expect a uniqueness result as in [8], 10.2.7.4, for the (weak) $\natural$-structures.

5.3. **Deligne's pairing.** Consider the canonical connection $\nabla^\sharp$ on $\mathcal{P}^\sharp$ defined in 5.2. Its curvature is an invariant 2-form on $G^\sharp \times G'^\sharp$; hence it gives an alternating pairing $R$ on

$$\mathrm{Lie}\,(G^\sharp \times G'^\sharp) = \mathrm{Lie}\,(G^\sharp) \oplus \mathrm{Lie}\,(G'^\sharp) = T_\sharp(M) \oplus T_\sharp(M^*)$$

with values in $k$. As the restrictions of $R$ to $\mathrm{Lie}\,(G^\sharp)$ and $\mathrm{Lie}\,(G'^\sharp)$ are trivial it holds

$$R(g_1 + g_2, g_1' + g_2') = \Phi(g_1, g_2') - \Phi(g_2, g_1')$$

with

$$\Phi\colon T_\sharp(M)\otimes T_\sharp(M^*)\to k. \tag{5.3.1}$$

If $M$ is a Deligne 1-motive, the pairing above coincides with the one in [8] 10.2.7.

We will see that $\Phi$ is perfect following the proof in [6], §4, for the classical case of Deligne 1-motives.

Recall the extensions in (3.1.3) for $M$ and $M^*$:

$$(5.3.2)\quad 0\longrightarrow \boldsymbol{\omega}_{G'}\xrightarrow{\ i\ } M^\sharp\xrightarrow{\ \rho\ } M\longrightarrow 0,\quad 0\longrightarrow \boldsymbol{\omega}_{G}\xrightarrow{\ i'\ } M^{*\sharp}\xrightarrow{\ \rho'\ } M^*\longrightarrow 0,$$

We denoted by $(\mathcal{P}_\rho,\nabla_2^\sharp)$ the $\natural$-extension of $M'$ by the multiplicative group over $G^\sharp$ that corresponds to the identity map on $G^\sharp$ *via* the functor $\mathcal{F}$ in 5.1.1. Similarly for $(\mathcal{P}_{\rho'},\nabla_1^\sharp)$.

5.3.3. **Lemma.** *Let $\alpha_{G'}$ be the invariant differential of $G'$ over $\boldsymbol{\omega}_{G'}$ that corresponds to the identity map on $\boldsymbol{\omega}_{G'}$. The restriction of $(\mathcal{P}_\rho,\nabla_2^\sharp)$ to $\boldsymbol{\omega}_{G'}$ via $i\colon \boldsymbol{\omega}_{G'}\to G^\sharp$ in (5.3.2) is isomorphic to the trivial extension of $M^*$ by the multiplicative group over $\boldsymbol{\omega}_{G'}$ equipped with the connection associated to $\alpha_{G'}$.*

*Proof.* See [6], 4.1. □

Changing the role of $M$ and $M^*$, denote by $\alpha_G$ the invariant differential of $G$ over $\boldsymbol{\omega}_G$ that corresponds to the identity map on $\boldsymbol{\omega}_G$. The restriction of $(\mathcal{P}_{\rho'},\nabla_1^\sharp)$ to $\boldsymbol{\omega}_G$ is isomorphic to the trivial extension of $G'$ by the multiplicative group over $\boldsymbol{\omega}_G$ equipped with the connection associated to $\alpha_G$. From [6], 4.2, we know that

5.3.4. **Lemma.** *The curvature of $\alpha_G$ provides a perfect pairing*

$$d\alpha_G\colon \omega_G\otimes \mathrm{Lie}\,(G)\longrightarrow k$$

*that is the usual duality.*

Hence the proof of Theorem 4.3 in *loc. cit.* works the same and we get

5.3.5. **Theorem.** *Let $M$ be a free $k$-1-motive. The pairing $\Phi$ in (5.3.1) is perfect. Moreover it fits in a diagram*

$$\begin{array}{ccccc}
\omega_{G'} & \otimes & \mathrm{Lie}\,(G') & \longrightarrow & k\\
\downarrow{\scriptstyle \iota} & & \uparrow{\scriptstyle g'} & & \\
\Phi\colon \mathrm{Lie}\,(G^\natural) & \otimes & \mathrm{Lie}\,(G'^\natural) & \longrightarrow & k\\
\downarrow{\scriptstyle g} & & \uparrow{\scriptstyle \iota'} & & \\
\mathrm{Lie}\,(G) & \otimes & \omega_G & \longrightarrow & k
\end{array}$$

*where the vertical homomorphisms come from (5.3.2) and the upper (resp. lower) pairing is the usual duality between the Lie algebra of $G'$ (resp. $G$) and the $k$-vector space of invariant differentials of $G'$ (resp. $G$).*

## 6. Sharp de Rham cohomology

We describe $H^1_{\sharp-\mathrm{dR}}(X) := T_\sharp(\mathrm{Pic}^+_a(X))$ in some meaningful cases, *i.e.*, when $X$ is proper or is a smooth algebraic $k$-scheme. Here $\mathrm{Pic}^+_a(X)$ and its Cartier dual $\mathrm{Alb}^-_a(X)$ are the Laumon 1-motives of the algebraic $k$-scheme $X$ constructed in [14], whence $\mathrm{Pic}^+_a(X)_{\text{ét}} = \mathrm{Pic}^+(X)$ and $\mathrm{Alb}^-_a(X)_{\text{ét}} = \mathrm{Alb}^-(X)$ were introduced in [5]. By construction, see 3.2, we then have that $H^1_{\sharp-\mathrm{dR}}(X)$ is sitting in an extension

$$0 \to H^1_{\mathrm{dR}}(X) \to H^1_{\sharp-\mathrm{dR}}(X)/V(\mathrm{Pic}) \to V(\mathrm{Alb}) \to 0$$

where we have set for $\mathrm{Pic}^+_a(X) = [F \to G]$

- $V(\mathrm{Pic}) :=$ the additive part, *i.e.*, (the Lie algebra of) the vector group $V(G)$ given by the maximal additive subgroup of $\mathrm{Pic}^0(\overline{X})$ for a suitable (singular) compactification $\overline{X}$ of $X$
- $V(\mathrm{Alb}) :=$ the infinitesimal part, *i.e.*, the Lie algebra $\mathrm{Lie}\, F^0$ that is just the dual of the corresponding Faltings-Wüstholz vector group in the Albanese $\mathrm{Alb}^-_a(X)$.

Here $V(\mathrm{Pic}) = 0$ if $X$ is smooth and $V(\mathrm{Alb}) = 0$ if $X$ is proper over $k$. Moreover, for $X$ smooth we have that $F_{\text{ét}} = \mathrm{Div}^0_Y(\overline{X})$ where $\overline{X}$ is a smooth proper compactification, $Y = \overline{X} - X$ is a normal crossing divisor, and $\mathrm{Lie}\, F^0$ is the $k$-vector space $\mathrm{Ker}(H^1(\overline{X}, \mathcal{O}_{\overline{X}}) \to H^1(X, \mathcal{O}_X))$, *i.e.*, is $\Gamma(X, \Omega^1_X)_{\mathrm{closed}}/d(\Gamma(X, \mathcal{O}_X))$ divided out by $\Gamma(\overline{X}, \Omega^1_{\overline{X}}(\log Y))$. For $X$ proper over $k$ it holds $F = 0$ and $G = \mathrm{Pic}^0(X)$ is the connected algebraic group given by the identity component of the representable fppf-sheaf $\mathrm{Pic}_{X/k}$ (see [14] for more details).

6.1. **Sharp extension of** $\mathrm{Pic}^+_a(X)$**.** We compute the sharp $\mathbb{G}_a$-extension of $\mathrm{Pic}^+_a(X)$ for $X$ proper or smooth.

For $X$ proper and $X_\bullet \to X$ a smooth proper hypercovering we obtain the semi-abelin quotient $\mathrm{Pic}^0(X)/V(\mathrm{Pic}) = \mathbb{P}\mathrm{ic}^0(X_\bullet)$ by [5], Lemma 5.1.2. In *loc. cit.* we also introduced the algebraic group $\mathbb{P}\mathrm{ic}^\natural(X_\bullet)$ given by isomorphism classes of triples $(\mathcal{L}, \nabla, \alpha)$ consisting of an invertible sheaf $\mathcal{L}$ on $X_0$, with an integrable connection $\nabla$, and an isomorphism $\alpha : d_0^*(\mathcal{L}, \nabla) \xrightarrow{\cong} d_1^*(\mathcal{L}, \nabla)$ satisfying the cocycle condition (here $d_0, d_1 : X_1 \to X_0$ are the face maps). There is a functorial isomorphism

$$\mathbb{P}\mathrm{ic}^\natural(X_\bullet) \cong \mathbb{H}^1(X_\bullet, \mathcal{O}^*_{X_\bullet} \xrightarrow{\mathrm{dlog}} \Omega^1_{X_\bullet}) \tag{6.1.1}$$

Define the group scheme $\mathrm{Pic}^\sharp(X) := \mathbb{P}\mathrm{ic}^\natural(X_\bullet) \times_{\mathbb{P}\mathrm{ic}(X_\bullet)} \mathrm{Pic}(X)$ by pull-back.

6.1.2. **Lemma.** *If $X$ is proper then* $\mathrm{Pic}^+_a(X)^\sharp = [0 \to \mathrm{Pic}^{\sharp,0}(X)]$.

*Proof.* Since $X_\bullet$ is smooth and proper over $k$, the semi-abelian variety $\mathbb{P}\mathrm{ic}^0(X_\bullet)$ is mapped to zero in $\mathbb{H}^1(X_\bullet, \Omega^1_{X_\bullet})$ *via* (6.1.1). We then have that $\mathbb{P}\mathrm{ic}^{\natural,0}(X_\bullet)$ is an extension of $\mathbb{P}\mathrm{ic}^0(X_\bullet)$ by $\mathbb{H}^0(X_\bullet, \Omega^1_{X_\bullet})$, *i.e.*, is the pull back of the inclusion $\mathbb{P}\mathrm{ic}^0 \hookrightarrow \mathbb{P}\mathrm{ic}$. This extension is the universal $\mathbb{G}_a$-extension of the semi-abelian scheme

$\mathbb{P}\mathrm{ic}^0(X_\bullet)$ by [5], Lemma 4.5.2. Since $V(\mathrm{Pic}) = \mathrm{Ker}(\mathrm{Pic}^{\sharp,0}(X) \twoheadrightarrow \mathbb{P}\mathrm{ic}^{\natural,0}(X_\bullet))$ we then get the following pullback diagram which, by (3.1.3), proves the assertion

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{H}^0(X_\bullet, \Omega^1_{X_\bullet}) \arrow[r] \arrow[d, equal] & \mathrm{Pic}^{\sharp,0}(X) \arrow[r] \arrow[d] & \mathrm{Pic}^0(X) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{H}^0(X_\bullet, \Omega^1_{X_\bullet}) \arrow[r] & \mathbb{P}\mathrm{ic}^{\natural,0}(X_\bullet) \arrow[r] & \mathbb{P}\mathrm{ic}^0(X_\bullet) \arrow[r] & 0 \end{tikzcd} \tag{6.1.3}$$

□

For $X$ smooth recall the algebraic group $\mathrm{Pic}^{\natural-\log}(X)$ given by isomorphism classes of pairs $(\mathcal{L}, \nabla^{\log})$ where $\mathcal{L}$ is a line bundle on $\overline{X}$ and $\nabla^{\log}$ is an integrable connection on $\mathcal{L}$ with log poles along $Y$. In [5], Lemma 2.6.2, we have seen that $\mathrm{Pic}^+_a(X)^\natural_{\text{ét}} = [\mathrm{Div}^0_Y(\overline{X}) \to \mathrm{Pic}^{\natural-\log,0}(X)]$. Actually, the universal $\mathbb{G}_a$-extension of $\mathrm{Pic}^+_a(X)$ exists (*cf.* the remark 2.3.3):

6.1.4. **Lemma.** *If $X$ is smooth then* $\mathrm{Pic}^+_a(X)^\natural = [F \to \mathrm{Pic}^{\natural-\log,0}(X) + \mathrm{Lie}\, F^0]$.

6.2. **Sharp de Rham over $\mathbb{C}$.** For $X$ over $\mathbb{C}$ we also have

$$H^1_{\sharp-\mathrm{dR}}(X) = T_{\oint}(\mathrm{Pic}^+_a(X))^\sharp$$

by 4.4.8. Let $X$ be a proper $\mathbb{C}$-scheme and $X_\bullet$ a smooth proper hypercovering as above. In this case, passing to the Lie algebra $\mathrm{Lie}\, \mathbb{P}\mathrm{ic}^{\natural,0}(X_\bullet) = H^1_{\mathrm{dR}}(X) \cong \mathbb{H}^1(X_\bullet, \mathbb{C}) \cong H^1(X, \mathbb{C})$ by the (simplicial) de Rham theorem and cohomological descent for the analytic topology, *cf.* [5], remark 2.6.3. We then obtain, *cf.* 4.4.6, an enriched Hodge structure *via* the following diagram of the Lie algebras of (6.1.3)

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{H}^0(X_\bullet, \Omega^1_{X_\bullet}) \arrow[r] \arrow[d, equal] & H^1_{\sharp-\mathrm{dR}}(X) \arrow[r, "u"] \arrow[d, "\pi"] & H^1(X, \mathcal{O}_X) \arrow[r] \arrow[d, "\pi_0"] & 0 \\ 0 \arrow[r] & \mathbb{H}^0(X_\bullet, \Omega^1_{X_\bullet}) \arrow[r] & H^1(X, \mathbb{C}) \arrow[r, "c"] & H^1(X_\bullet, \mathcal{O}_{X_\bullet}) \arrow[r] & 0 \end{tikzcd} \tag{6.2.1}$$

where $H^1(X, \mathbb{C})/F^1_{Hodge} = H^1(X_\bullet, \mathcal{O}_{X_\bullet})$, $\mathbb{H}^0(X_\bullet, \Omega^1_{X_\bullet}) = F^1_{Hodge}$ and $V(\mathrm{Pic}) = \mathrm{Ker}(H^1(X, \mathcal{O}_X) \twoheadrightarrow H^1(X_\bullet, \mathcal{O}_{X_\bullet}))$. If $X$ is smooth we have $\mathrm{Lie}\, \mathrm{Pic}^{\natural-\log,0}(X) = H^1(X, \mathbb{C})$ by [5], 2.6.4. We then obtain:

6.2.2. **Proposition.** *For $X$ over $\mathbb{C}$ we have $H^1_{\sharp-\mathrm{dR}}(X) \cong H^1(X, \mathbb{C}) \oplus V$ where $V = V(\mathrm{Pic})$ if $X$ is proper and $V = V(\mathrm{Alb})$ if $X$ is smooth.*

*Proof.* It follows from the previous lemmas in 6.1 and the above discussion. □

6.2.3. **Corollary.** *If $X$ is a proper $\mathbb{C}$-scheme and $H^1(X, \mathbb{Z}) = 0$ then $H^1_{\sharp-\mathrm{dR}}(X) = H^1(X, \mathcal{O}_X)$.*

Let $X$ be now a proper (reduced) variety over $\mathbb{C}$ and consider, following [7], the naive analytic de Rham complex $\Omega^{\bullet}_X$ on $X$ itself. The resulting cohomology $\mathbb{H}^n(X, \Omega^{\bullet}_X)$ is considered in [7] as part of one possible enriched Hodge structure associated to $X$ and $n$. Actually for $'\Omega^{\bullet}_X := \Omega^{\bullet}_X / F^{\dim(X)+1}\Omega^{\bullet}_X$ and $''\Omega^{\bullet}_X := \Omega^{\bullet}_X/\mathrm{tors}$ we have $\Omega^{\bullet}_X \twoheadrightarrow {}'\Omega^{\bullet}_X \twoheadrightarrow {}''\Omega^{\bullet}_X$. Moreover

$$\mathbb{H}^n(X, \Omega^{\bullet}_X) \to \mathbb{H}^n(X,\ '\Omega^{\bullet}_X) \to \mathbb{H}^n(X,\ ''\Omega^{\bullet}_X) \to \mathbb{H}^n(X_{\bullet}, \Omega^{\bullet}_{X_{\bullet}}) \cong H^n(X, \mathbb{C})$$

yields three different enriched Hodge structures associated to $X$ and $n$, see [7], 2.1 and 2.2. For $n = 1$ we want to compare them with $H^1_{\sharp-\mathrm{dR}}(X)$. We clearly have, by construction, a commutative diagram with exact rows

$$\text{(6.2.4)} \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{H}^0(X,\ ''\Omega^{\bullet \geq 1}_X) & \longrightarrow & \mathbb{H}^1(X,\ ''\Omega^{\bullet}_X) & \longrightarrow & H^1(X, \mathcal{O}_X) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{H}^0(X_{\bullet}, \Omega^1_{X_{\bullet}}) & \longrightarrow & H^1(X, \mathbb{C}) & \longrightarrow & H^1(X_{\bullet}, \mathcal{O}_{X_{\bullet}}) & \longrightarrow & 0 \end{array}$$

providing a canonical comparison map

$$\rho'' : \mathbb{H}^1(X,\ ''\Omega^{\bullet}_X) \to H^1_{\sharp-\mathrm{dR}}(X)$$

and similarly by composition $\rho' : \mathbb{H}^1(X,\ '\Omega^{\bullet}_X) \to H^1_{\sharp-\mathrm{dR}}(X)$ and $\rho : \mathbb{H}^1(X, \Omega^{\bullet}_X) \to H^1_{\sharp-\mathrm{dR}}(X)$. In fact, just observe that $H^1_{\sharp-\mathrm{dR}}(X)$ is given by the pull-back diagram (6.2.1) and therefore *any* such enriched Hodge structure associated to $X$ and $n = 1$ maps to it.

6.2.5. **Proposition.** *Let $\rho^{\dagger}$ denote any comparison map $\rho, \rho'$ or $\rho''$ corresponding to ${}^{\dagger}\Omega^{\bullet}_X$ which denotes $\Omega^{\bullet}_X$, $'\Omega^{\bullet}_X$ and $''\Omega^{\bullet}_X$ respectively. The decorated comparison map $\rho^{\dagger}$ is*

(i) *surjective if and only if the boundary map $H^1(X, \mathcal{O}_X) \to \mathbb{H}^1(X, {}^{\dagger}\Omega^{\bullet \geq 1}_X)$ is zero and $\mathbb{H}^0(X, {}^{\dagger}\Omega^{\bullet \geq 1}_X) \twoheadrightarrow \mathbb{H}^0(X_{\bullet}, \Omega^1_{X_{\bullet}})$ is surjective;*

(ii) *injective if and only if the map $\mathbb{H}^0(X, {}^{\dagger}\Omega^{\bullet \geq 1}_X) \hookrightarrow \mathbb{H}^0(X_{\bullet}, \Omega^1_{X_{\bullet}})$ is an inclusion.*

*The map $\rho^{\dagger}$ is then an isomorphism if and only if both conditions hold.*

*Proof.* Comparing the diagram (6.2.1) and the decorated version of (6.2.4) observe that the top exact sequence in the latter continues on the right with the mentioned boundary map. □

6.2.6. **Remark.** If $\pi_1(X) = 0$ then $\rho^{\dagger} : \mathbb{H}^1(X, {}^{\dagger}\Omega^{\bullet}_X) \to H^1(X, \mathcal{O}_X) = H^1_{\sharp-\mathrm{dR}}(X)$. For example, if we take the curve considered in 2.3 of [7] then $\rho''$ is an isomorphism. Note that it seems puzzling to study the geometric meaning of these

conditions. In general, we just have that

$$\mathbb{H}^0(X_\bullet, \Omega^1_{X_\bullet}) = \operatorname{Ker} H^0(X_0, \Omega^1_{X_0}) \stackrel{d_0^*-d_1^*}{\longrightarrow} H^0(X_1, \Omega^1_{X_1})$$

for the components $X_0, X_1$ of $X_\bullet$ while

$$\mathbb{H}^0(X, {}^\dagger\Omega^{\bullet\geq 1}_X) = \operatorname{Ker} H^0(X, {}^\dagger\Omega^1_X) \to H^0(X, {}^\dagger\Omega^2_X)$$

For $\dim(X) = 1$, by choosing $X_\bullet$ in such a way that $X_0$ is the normalization of $X$ and $X_1$ is 0-dimensional, we have $\mathbb{H}^0(X_\bullet, \Omega^1_{X_\bullet}) = H^0(X_0, \Omega^1_{X_0})$, $\mathbb{H}^0(X, {}'\Omega^{\bullet\geq 1}_X) = H^0(X, \Omega^1_X)$ and further $\mathbb{H}^0(X, {}''\Omega^{\bullet\geq 1}_X) = H^0(X, \Omega^1_X/\mathrm{tors})$, *e.g.*, the injectivity of $\rho'$ means that $H^0(X, \Omega^1_X)$ injects into $H^0(X_0, \Omega^1_{X_0})$.

## Appendix A.

In this section we recall some facts and results on (formal) $k$-groups needed in the paper. The characteristic of the field $k$ is zero.

A.1. **Vector groups.** Let $\mathcal{E}$ be a free $k$-module and $\mathcal{E}^\vee = \operatorname{Hom}(\mathcal{E}, k)$ its dual. Denote by $E = \operatorname{Spec}(k[\mathcal{E}^\vee])$ the $k$-vector group associated to $\mathcal{E}$ where

$$k[\mathcal{E}^\vee] = \operatorname{Sym}(\mathcal{E}^\vee) = k \oplus \mathcal{E}^\vee \oplus S^2(\mathcal{E}^\vee) \oplus \dots$$

Its completion at the origin is $\widehat{E} = \operatorname{Spf}(k[[\mathcal{E}^\vee]])$ where $k[[\mathcal{E}^\vee]]$ means the infinite product

$$k \times \mathcal{E}^\vee \times S^2(\mathcal{E}^\vee) \times S^3(\mathcal{E}^\vee) \times \dots$$

with the multiplication induced by that of $\operatorname{Sym}(\mathcal{E}^\vee)$.

A.1.1. **Remark.** Starting with $\widehat{E}$ we can recover $E$ *via* $\mathcal{E} = \operatorname{Lie}(\widehat{E})$ (*cf.* [11], $\mathrm{VII}_B$ 3.3).

A.2. **On Cartier duals.** Let $F = \operatorname{Spf}(A)$ be a connected formal $k$-group. Its Cartier dual[2] $F^*$ is defined as $\operatorname{Spec}(A^*)$ with $A^* := \operatorname{Hom}_{\mathrm{cont}}(A, k)$ where $k$ is endowed with the discrete topology. For example, if $A = k[[x]]$, any continuous $k$-linear map $f\colon A \to k$ factors through $k[[x]]/(x^n)$ because $f^{-1}(0)$ has to be open in $A$. Set $f(1) = a_0, f(x^i) = a_i$; then $f$ is uniquely determined by the polynomial $\sum_i a_i(x^*)^i$. Hence $A^* = k[x^*]$. Observe that $x^*$ corresponds to the $k$-linear map sending $1 \mapsto 0$, $x \mapsto 1$ and $x^i \mapsto 0$ for $i > 1$. Similarly $k[[x_1, \dots, x_n]]^* = k[x_i^*, \dots, x_n^*]$.

The duality between $F^0$ and $F^{0*}$ provides also a duality on Lie algebras.

A.2.1. **Lemma.** *Let $F = \operatorname{Spf}(k[[x_1, \cdots, x_n]])$ be a connected formal $k$-group and $F^* = \operatorname{Spec}(k[[x_1, \cdots, x_n]]^*)$ its Cartier dual. There is a canonical duality between $\operatorname{Lie}(F)$ and $\operatorname{Lie}(F^*)$.*

[2]It is denoted by $\mathbb{D}(F)$ in [10].

*Proof.* The Lie algebra of $F$ corresponds to the $k$-linear maps

$$k[[x_1, \cdots, x_n]] \to k[\epsilon]/(\epsilon^2)$$

such that $a \mapsto a$ for $a \in k$, $x_i \mapsto b_i\epsilon$ with $b_i \in k$ and $x_ix_j \mapsto 0$, hence to the $k$-linear polynomial $\sum_{i=1}^n b_ix_i^*$. Recall now that

$$F^* = \mathrm{Spec}(k[[x_1, \cdots, x_n]]^*) = \mathrm{Spec}(k[x_1^*, \ldots, x_n^*]).$$

The Lie algebra of $F^*$ is the $k$-module of $k$-linear polynomials in the $n$-variables $x_i^{**}$. Hence there is a canonical pairing $\mathrm{Lie}\,(F) \otimes \mathrm{Lie}\,(F^*) \to k$ sending $x_i^* \otimes x_j^{**}$ to $\delta_{ij}$ that does not depend on the choice of the basis $x_i$. □

A.3. **Formal completion at the origin.** Let $G$ be a connected algebraic $k$-group. The connected formal $k$-group associated to $\mathrm{Lie}\,(G)$ is canonically isomorphic to the formal completion at the origin of $G$. Indeed, let $x_1, \cdots, x_g$ be free generators of $\mathrm{Lie}\,(G)^\vee$ over $k$. The associated formal $k$-group is $\mathrm{Spf}(k[[x_1, \cdots, x_g]])$; moreover, as $\mathrm{Lie}\,(G)^\vee$ is canonically isomorphic to the $k$-module of invariant differentials on $G$ and hence to $m/m^2$ with $m$ the maximal ideal of $\mathcal{O}_{G,e}$, we could think $\{x_i\}_i$ as a basis of the $k$-module $m/m^2$. Now, the formal completion at the origin of $G$ is the formal spectrum of

$$\varprojlim_n \mathcal{O}_{G,e}/m^n = k \times m/m^2 \times m^2/m^3 \times \cdots = k[[x_1, \ldots, x_g]].$$

As a consequence of A.2.1 we get then

A.3.1**. Lemma.** *Let $G$ be an algebraic $k$-group. The Cartier dual of its formal completion $\widehat{G}$ is canonically isomorphic to $\boldsymbol{\omega}_G = \mathrm{Spec}(k[\mathrm{Lie}\,(G)])$.*

A.4. **Homomorphisms and extensions.** We defined “strongly exact” sequences in $\mathcal{M}_1^{\mathrm{a,eff}}$ as exact sequences of complexes in $\mathbf{Ab}/k$. It is immediate to prove the following:

A.4.1**. Lemma.** *Il $M$ is an effective $k$-1-motive and $E$ is any sheaf in $\mathbf{Ab}/k$ the morphism*

$$\mathrm{Ext}_{C^{[-1,0]}(\mathbf{Ab}/k)}(M, E) \longrightarrow \mathrm{Hom}_{D^b(\mathbf{Ab}/k)}(M, E[1])$$

*is an isomorphism.*

Now, $\mathcal{M}_1^{\mathrm{a,eff}}$ is an exact subcategory of $\mathcal{M}_1^{\mathrm{a}}$ and strongly exact sequences of effective 1-motives are exact in $\mathcal{M}_1^{\mathrm{a}}$. The converse is not true in general. However, any exact sequence in $\mathcal{M}_1^{\mathrm{a}}$ can be represented by a strongly exact sequence (1.3, *cf.* [4] for the classical case). Furthermore, $\mathbb{G}_a$-extensions of 1-motives are isomorphic to strongly exact extensions.

A.4.2**. Proposition.** *Let $M$ be an effective $k$-1-motive and $W$ a $k$-vector group. Any isomorphism class of extensions of $M$ by $W$ in $\mathcal{M}_1^{\mathrm{a}}$ contains a strongly exact extension of $M$ by $W$ and the canonical map*

$$\mathrm{Ext}_{C^{[-1,0]}(\mathbf{Ab}/k)}(M, W) \longrightarrow \mathrm{Ext}_{\mathcal{M}_1^{\mathrm{a}}}(M, W)$$

*is an isomorphism.*

*Proof.* The injectivity follows immediately from the fact that any q.i. between 1-motives $[F \to G_i]$, $i = 1, 2$, is an isomorphism. For the surjectivity, let $\widetilde{G} \to [F_1 \to G_1] \xrightarrow{f} M$ be effective morphisms that provide an extension in $\mathcal{M}_1^{\mathrm{a}}$. It means that $\iota\colon \widetilde{G} \to G_1$ is a monomorphism and $f$ induces epimorphisms $f_F\colon F_1 \to F$ on the formal groups and $f_G\colon G_1 \to G$ on the algebraic $k$-groups. Moreover, $\mathrm{Ker}(f_F) = \mathrm{Ker}(f_G)/\,\mathrm{Im}(\iota)$. If now $\widetilde{G}$ is a vector group $W$, one deduces easily that $W$ is the kernel of the restriction of $f$ to $V(G_1)$ and that any extension $[F_1 \to G_1]$ of $M$ by $W$ in $\mathcal{M}_1^{\mathrm{a}}$ is isomorphic to the extension $[F \to G_1/\,\mathrm{Ker}(f_F)]$ of $M$ by $W$ in $\mathcal{M}_1^{\mathrm{a,eff}}$. □

A.4.3. **Lemma.** *Let $F$ be a formal $k$-group. Then $\mathrm{Hom}(F, \mathbb{G}_a)$ is a free $k$-module of finite rank.*

*Proof.* For the connected part, it is sufficient to consider the case $F^0 = \widehat{\mathbb{G}}_a$. Now, $\mathrm{Hom}(\widehat{\mathbb{G}}_a, \mathbb{G}_a) = \mathrm{Hom}(\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a) = k$. For $F$ étale and free, $\mathrm{Hom}(F, \mathbb{G}_a) = \mathrm{Lie}\,(T')$ where $T'$ is the Cartier dual of $F$. For $F_{\mathrm{tor}}$ one has $\mathrm{Hom}(F_{\mathrm{tor}}, \mathbb{G}_a) = 0$. □

A.4.4. **Lemma.** *Let $G$ be a connected algebraic $k$-group and $F$ a formal $k$-group. Then* $\mathrm{Hom}(G, F) = 0$.

*Proof.* As $G$ is connected, $\mathrm{Hom}(G, F_{\text{ét}}) = 0$. It remains to prove that $\mathrm{Hom}(G, \widehat{\mathbb{G}}_a) = 0$. Any morphism $f\colon L \to \widehat{\mathbb{G}}_a$, with $L$ a linear $k$-group, is trivial because $L$ is reduced. Suppose then $G = A$ an abelian variety and let $f\colon A \to \widehat{\mathbb{G}}_a$ be a morphism. The induced morphism $A \to \mathbb{G}_a$ is trivial. Moreover for any $k$-algebra $C$, $\widehat{\mathbb{G}}_a(C) = \mathrm{Nil}(C)$ injects into $\mathbb{G}_a(C) = C$. Hence also $f$ is trivial. □

A.4.5. **Lemma.** *Let $F$ be a formal $k$-group without torsion and $G$ an algebraic connected $k$-group. Then* $\mathrm{Ext}_{\mathbf{Ab}/k}(G, F) = 0$.

*Proof.* For $F = F_{\text{ét}}$ see [16], 2.3.2. For $F = F^0$ we reduce to the case $F = \widehat{\mathbb{G}}_a$. Observe that

$$\mathrm{Ext}_{\mathbf{Ab}/k}(G, \widehat{\mathbb{G}}_a) = \mathrm{Ext}_{\mathbf{Ab}/k}(G, \underline{\mathrm{Hom}}(\mathbb{G}_a, \mathbb{G}_m)) = \mathrm{Biext}^1(G, \mathbb{G}_a; \mathbb{G}_m)$$

where we use Cartier duality for the first isomorphism and the exact sequence

$$\mathrm{Ext}_{\mathbf{Ab}/k}(P, \underline{\mathrm{Hom}}(Q, \mathbb{G}_m)) \to \mathrm{Biext}^1(P, Q; \mathbb{G}_m) \to \mathrm{Hom}(P, \underline{\mathrm{Ext}}(Q, \mathbb{G}_m))$$

(*cf.* [12], VIII, 1.1.4) with $P = G$, $Q = \mathbb{G}_a$ for the second. Moreover

$$\mathrm{Biext}^1(G, \mathbb{G}_a; \mathbb{G}_m) = \mathrm{Biext}^1(\mathbb{G}_a, G; \mathbb{G}_m) = 0$$

because of [12], VII, 3.6.5 & VIII, 4.6. □

A.4.6. **Lemma.** *Let $F$ be a connected formal $k$-group. Then $\mathrm{Ext}_{\mathbf{Ab}/C}(F, \mathbb{G}_a) = 0$ for any $k$-algebra $C$ and $\underline{\mathrm{Ext}}(F, \mathbb{G}_m) = 0$.*

*Proof.* We reduce to the case $F = \widehat{\mathbb{G}}_a$. Suppose given an extension

$$0 \to \mathbb{G}_a \to H \to \widehat{\mathbb{G}}_a \to 0$$

over $k$. We show that it is trivial. Denote by $H_n$ the pull-back of $H$ to the $n$-infinitesimal neighborhood $\mathbb{G}_{a,n} = \mathrm{Spec}(k[x]/(x^{n+1}))$ of $\widehat{\mathbb{G}}_a$. The scheme $H_n$ is an $\mathbb{G}_a$-torsor over $\mathbb{G}_{a,n}$ and hence trivial. In particular $H_n$ is smooth over $\mathbb{G}_{a,n}$. Recalling now that $\mathbb{G}_{a,n-1} \to \mathbb{G}_{a,n}$ is a closed immersion with square zero ideal, the lifting property of smooth morphisms permits to construct a tower of compatible sections $s_n \colon \mathbb{G}_{a,n} \to H_n \to H$ and hence compatible "factor sets" $\gamma_n \colon \mathbb{G}^2_{a,n} \to \mathbb{G}_a$ defined as

$$\gamma_n(a,b) := s_{2n}(a+b) - s_n(a) - s_n(b).$$

We may suppose that $\gamma$ is normalized, *i.e.*, $s_n(0) = 0 \in \mathbb{G}_a(k)$. We may summarize this fact saying that we have a morphism $\gamma\colon \widehat{\mathbb{G}}_a \to \mathbb{G}_a$ as contravariant functor from $\mathbf{Aff}/k$ to the category of sets satisfying the usual properties of a factor set. Let $P = \sum a_{ij}x_1^i x_2^j$ be the associated power series in $k[[x_1, x_2]]$. As $\gamma_n(0,b) = \gamma_n(a,0) = s_n(0) = 0$ the polynomial $\gamma_n^*(x)$ (that is $P$ truncated at the $n$th powers) is divisible by $x_1x_2$ and $\gamma_n$ factors through $\mathbb{G}_{a,n}$. It provides then a "factor set" $\widehat{\gamma}\colon \widehat{\mathbb{G}}^2_a \to \widehat{\mathbb{G}}_a$ and $H$ is the push-out along $\widehat{\mathbb{G}}_a \xrightarrow{\iota} \mathbb{G}_a$ of an extension $E$ of $\widehat{\mathbb{G}}_a$ by itself. As any extension of connected formal $k$-groups is trivial, $E$ is trivial and hence the same is $H$.

Let now $C$ be a $k$-algebra and $H$ an extension of $\widehat{\mathbb{G}}_a$ by $\mathbb{G}_a$ over $C$. We can repeat the above construction getting "factor sets" $\gamma_n$ over $C$ determined by a power series $P = \sum a_{ij}x_1^i x_2^j$ in $C[[t]]$. In order to see that $H$ is trivial, one is reduced to see that $P$ can be written as $h(x_1+x_2) - h(x_1) - h(x_2)$ for a suitable power series $h(t) = \sum b_n t^n$ with coefficients in $C$. This is possible if

$$\frac{a_{ij}}{\binom{i+j}{i}} = \frac{a_{mn}}{\binom{m+n}{m}} \quad \text{whenever} \quad i+j = m+n. \tag{A.4.7}$$

This fact can be deduced comparing

$$\sum a_{ij}x_1^i x_2^j \quad \text{and} \quad \sum b_n(x_1+x_2)^n - \sum b_n x_1^n - \sum b_n x_2^n.$$

Using now the property of factor sets

$$\gamma(a,b) - \gamma(a, b+c) = \gamma(b,c) - \gamma(a+b,c)$$

we get

$$\sum a_{ij}x^i y^j - \sum a_{ij}x^i(y+z)^j = \sum a_{ij}y^i z^j - \sum a_{ij}(x+y)^i z^j, \quad \text{or}$$

$$\sum a_{ij}x^i[y^j - (y+z)^j] = \sum a_{ij}[y^i - (x+y)^i]z^j. \tag{A.4.8}$$

Now $a_{ij}\binom{j}{s}$ is the coefficient of the term $-x^i y^{j-s}z^s$ while $a_{mn}\binom{m}{q}$ is the coefficient of the term $-x^{m-q}y^q z^n$. Observe that a monomial $x^a y^b z^c$ occurs once on the left for $i = a$ and $j = b+c$ and on the right for $m = a+b$ and $n = c$. Considering

the same monomial on the right and on the left, the indices satisfy the following relations: $i+j=m+n$, $s=n$ and $q=j-n$. The A.4.8 implies that

$$a_{ij}\binom{j}{n}=a_{mn}\binom{m}{j-n}.$$

As

$$\binom{j}{n}\binom{i+j}{i}=\binom{m}{j-n}\binom{m+n}{m}$$

condition A.4.7 is satisfies.

Let now $H$ be an extension of $\widehat{\mathbb{G}}_a$ by $\mathbb{G}_m=\mathrm{Spec}(k[z,z^{-1}])$ over a $k$-algebra $C$. Let $H_n$ be the pull-back of $H$ to $\mathbb{G}_{a,n}$. Étale locally on $C$, $H_n$ is a trivial $\mathbb{G}_m$-torsor. We may suppose that it is trivial on $C$. Proceed then as above constructing sections $s_n\colon \mathbb{G}_{a,n}\to H$ and normalized "factor sets" $\gamma_n$ such that $\gamma_n^*(z-1)$ is divisible by $x_1x_2$. Hence $\gamma_n$ factors through $\mathrm{Spec}(C[z,z^{-1}]/(z-1)^n)$ and we get a factor set $\gamma\colon \widehat{\mathbb{G}}_a^2\to\widehat{\mathbb{G}}_m$ over $C$. As $\widehat{\mathbb{G}}_m$ is isomorphic to $\widehat{\mathbb{G}}_a$, we proceed as done for $\mathbb{G}_a$, showing that $\gamma$ is equivalent to the trivial factor set. □

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